PHPE 400
Individual and Group Decision Making

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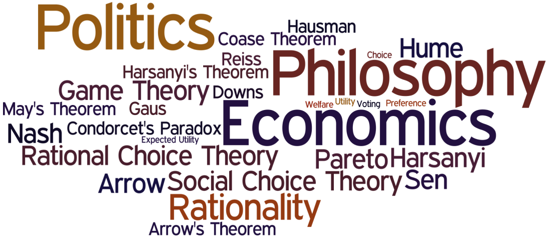

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What additional properties should a *rational preference* (P, I) on $\mathcal{L}(X)$ satisfy?

Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b) and c is another item.

How *should* the decision maker rank the lotteries

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1. $L_1 \; P \; L_2$: The decision maker should strictly prefer L_1 to L_2.
2. $L_2 \; P \; L_1$: The decision maker should strictly prefer L_2 to L_1.
3. $L_1 \; I \; L_2$: The decision maker should be indifferent between L_1 and L_2.
4. There is not enough information to answer this question.

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4. There is not enough information to answer this question.

Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b) and c is another item.

Then, a *rational* decision maker will have the following preferences:

1. $(0.6 \cdot a + 0.4 \cdot c) \; P \; (0.6 \cdot b + 0.4 \cdot c)$.
2. $(0.6 \cdot a + 0.4 \cdot b) \; P \; (0.4 \cdot a + 0.6 \cdot b)$.

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Neither of these preferences can be inferred if all you know is that the decision maker's preferences over lotteries satisfies transitivity and completeness.

A **rational** preference over lotteries involves more than the assumption that the decision maker's preferences are transitive and complete:

1. Independence axiom
2. Continuity axiom

Independence

For all $L_1, L_2, L_3 \in \mathcal{L}$ and $0 < p \leq 1$,

$$L_1 \; P \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3).$$

$$L_1 \; I \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3).$$

Independence

For all $L, L', L'' \in \mathcal{L}$ and $0 < p \leq 1$,

$$L \; P \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; P \; (p \cdot L' + (1-p) \cdot L'').$$

$$L \; I \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; I \; (p \cdot L' + (1-p) \cdot L'').$$

Independence

For all $\color{blue}\blacksquare$, $\color{red}\blacksquare$, $\color{green}\blacksquare$ $\in \mathcal{L}$ and $0 < p \leq 1$,

$$\color{blue}\blacksquare\ \color{black}P\ \color{red}\blacksquare \quad \text{if, and only if,} \quad \color{black}(p \cdot \color{blue}\blacksquare \color{black}+ (1-p) \cdot \color{green}\blacksquare\color{black})\ P\ (p \cdot \color{red}\blacksquare \color{black}+ (1-p) \cdot \color{green}\blacksquare\color{black}).$$

$$\color{blue}\blacksquare\ \color{black}I\ \color{red}\blacksquare \quad \text{if, and only if,} \quad \color{black}(p \cdot \color{blue}\blacksquare \color{black}+ (1-p) \cdot \color{green}\blacksquare\color{black})\ I\ (p \cdot \color{red}\blacksquare \color{black}+ (1-p) \cdot \color{green}\blacksquare\color{black}).$$

Suppose that a decision maker has the following preference:

$$(1 \cdot \$2000) \quad P \quad (0.6 \cdot \$3000 + 0.4 \cdot \$0)$$

Assuming that the decision maker satisfies the Independence Axiom, what is the decision maker's preference between the following two lotteries?

- $L_1 = 0.5 \cdot \$2000 + 0.5 \cdot \$0$
- $L_2 = 0.3 \cdot \$3000 + 0.7 \cdot \$0$

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We can show that the decision maker must strictly prefer L_1 to L_2

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iff (Independence)

$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

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$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

iff (Simplifying lotteries)

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad ((0.5 \times 0.6) \cdot \$3000 + (0.5 \times 0.4 + 0.5) \cdot \$0)$$

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Example

Consider the set of all lotteries over $X = \{a, b\}$.

Suppose that Ann prefers lotteries that are closer to 50-50. For example,

- $(0.5 \cdot a + 0.5 \cdot b) \;\; P \;\; (0.25 \cdot a + 0.75 \cdot b)$
- $(0.75 \cdot a + 0.25 \cdot b) \;\; I \;\; (0.25 \cdot a + 0.75 \cdot b)$
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- $(0.25 \cdot a + 0.75 \cdot b) \;\; P \;\; (1 \cdot a)$

We can view Ann as assigning a value to any lottery as follows:

$$\text{A lottery } r \cdot a + (1 - r) \cdot b \text{ is valued at } -|r - \tfrac{1}{2}|.$$

Then, Ann ranks lotteries by assigning a value to the lotteries and ranking them according to the values.