PHPE 400
Individual and Group Decision Making

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A **rational** preference over lotteries involves more than the assumption that the decision maker's preferences are transitive and complete:

1. Independence axiom
2. Continuity axiom

Independence

For all $L_1, L_2, L_3 \in \mathcal{L}$ and $0 < p \leq 1$,

$$L_1 \; P \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3).$$

$$L_1 \; I \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3).$$

Suppose that a decision maker has the following preference:

$$(1 \cdot \$2000) \quad P \quad (0.6 \cdot \$3000 + 0.4 \cdot \$0)$$

Assuming that the decision maker satisfies the Independence Axiom, what is the decision maker's preference between the following two lotteries?

- $L_1 = 0.5 \cdot \$2000 + 0.5 \cdot \$0$
- $L_2 = 0.3 \cdot \$3000 + 0.7 \cdot \$0$

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We can show that the decision maker must strictly prefer L_1 to L_2

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iff (Independence)

$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

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iff (Simplifying lotteries)

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad ((0.5 \times 0.6) \cdot \$3000 + (0.5 \times 0.4 + 0.5) \cdot \$0)$$

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Independence

For all ${\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{green}\blacksquare} \in \mathcal{L}$ and $0 < p \leq 1$,

$${\color{blue}\blacksquare}\ P\ {\color{red}\blacksquare} \quad \text{if, and only if,} \quad (p \cdot {\color{blue}\blacksquare} + (1-p) \cdot {\color{green}\blacksquare})\ P\ (p \cdot {\color{red}\blacksquare} + (1-p) \cdot {\color{green}\blacksquare}).$$

$${\color{blue}\blacksquare}\ I\ {\color{red}\blacksquare} \quad \text{if, and only if,} \quad (p \cdot {\color{blue}\blacksquare} + (1-p) \cdot {\color{green}\blacksquare})\ I\ (p \cdot {\color{red}\blacksquare} + (1-p) \cdot {\color{green}\blacksquare}).$$

Example

Consider the set of all lotteries over $X = \{a, b\}$.

Suppose that Ann prefers lotteries that are closer to 50-50. For example,

- $(0.5 \cdot a + 0.5 \cdot b)\ \ P\ \ (0.25 \cdot a + 0.75 \cdot b)$
- $(0.75 \cdot a + 0.25 \cdot b)\ \ I\ \ (0.25 \cdot a + 0.75 \cdot b)$
- $(0.25 \cdot a + 0.75 \cdot b)\ \ P\ \ (1 \cdot a)$

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- $(0.25 \cdot a + 0.75 \cdot b) \;\; P \;\; (1 \cdot a)$

We can view Ann as assigning a value to any lottery as follows:

$$\text{A lottery } r \cdot a + (1 - r) \cdot b \text{ is valued at } -|r - \tfrac{1}{2}|.$$

Then, Ann ranks lotteries by assigning a value to the lotteries and ranking them according to the values.

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- Do Ann’s preferences satisfy completeness? Yes.
- Do Ann’s preferences satisfy the Independence Axiom? No.

Ann's preferences violates the Independence Axiom since she has the following preferences:

- $(0.5 \cdot a + 0.5 \cdot b) \;\; P \;\; (1 \cdot a)$
- It is **not** the case that

$$(0.5 \cdot \;\; (0.5 \cdot a + 0.5 \cdot b) \;\; + 0.5 \cdot \;\; b \;) \; P \; (0.5 \cdot \;\; (1 \cdot a) \;\; + 0.5 \cdot \;\; b \;).$$

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$$(0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b)\ P\ (0.5 \cdot (1 \cdot a) + 0.5 \cdot b).$$

This is because

$$\begin{aligned}(0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b) &= (0.25 \cdot a + 0.75 \cdot b)\\ (0.5 \cdot (1 \cdot a) + 0.5 \cdot b) &= (0.5 \cdot a + 0.5 \cdot b)\end{aligned}$$

And so, $(0.5 \cdot a + 0.5 \cdot b)\ P\ (0.25 \cdot a + 0.75 \cdot b)$ since the value of $(0.5 \cdot a + 0.5 \cdot b)$ is 0 and $(0.25 \cdot a + 0.75 \cdot b)$ is -0.25 and $-0.25 < 0$. Hence, it is not the case that $(0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b)\ P\ (0.5 \cdot (1 \cdot a) + 0.5 \cdot b)$.

Violating Independence

A decision maker **does not** satisfy the Independence Axiom when there are lotteries L_1, L_2, L_3 and a number p such that $0 < p \leq 1$ such that at least one of the following is true:

1. $L_1 \; P \; L_2$, but it is not the case that $(p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3)$;

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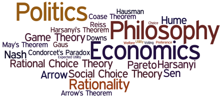

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2. $(p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3)$, but it is not the case that $L_1 \; P \; L_2$;
3. $L_1 \; I \; L_2$, but it is not the case that $(p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3)$; or
4. $(p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3)$, but it is not the case that $L_1 \; I \; L_2$;