PHPE 400
Individual and Group Decision Making

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Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \to \mathbb{R}$ is a utility function on X.

The **expected utility** of a lottery $L = p_1 \cdot x_1 + \cdots + p_n \cdot x_n$ with respect to u is defined as follows:

$$\begin{aligned} EU(p_1 \cdot x_1 + \cdots + p_n \cdot x_n, u) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\ &= \textstyle\sum_{i=1}^{n} p_i \times u(x_i) \end{aligned}$$

Expected Utility: Example

Let $X = \{a, b, c\}$ and $u : X \to \mathbb{R}$ where $u(a) = 2$, $u(b) = 4$, and $u(c) = 0$. Then,

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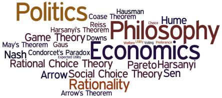

Let $X = \{a, b, c\}$ and $u : X \to \mathbb{R}$ where $u(a) = 2$, $u(b) = 4$, and $u(c) = 0$. Then,

$EU(0.25 \cdot a + 0.25 \cdot b + 0.5 \cdot c, u)$

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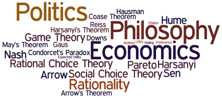

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Taking Stock

- Expected value and expected utility (with respect to some utility function) are often used to compare lotteries.
- Comparing lotteries by their expected values may result in a different ranking than comparing lotteries by their expected utility with respect to some utility function.
- To calculate the expected utility of a lottery we need the decision maker's utility function on the outcomes.

Comparing Lotteries

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Suppose that X is a set of outcomes and $\mathcal{L}(X)$ is the set of all lotteries over X.

Are there additional properties beyond completeness and transitivity that a *rational preference* (P, I) on $\mathcal{L}(X)$ should satisfy?

Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b) and c is another item.

How *should* the decision maker rank the lotteries

$$L_1 = 0.6 \cdot a + 0.4 \cdot c \quad \text{and} \quad L_2 = 0.6 \cdot b + 0.4 \cdot c?$$

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1. $L_1 \; P \; L_2$: The decision maker should strictly prefer L_1 to L_2.
2. $L_2 \; P \; L_1$: The decision maker should strictly prefer L_2 to L_1.
3. $L_1 \; I \; L_2$: The decision maker should be indifferent between L_1 and L_2.
4. There is not enough information to answer this question.

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4. There is not enough information to answer this question.

Suppose that the decision maker is rational and has the preference $a \, P \, b$ (the decision maker strictly prefers a to b) and c is another item.

Then, a *rational* decision maker will have the following preferences:

1. $(0.6 \cdot a + 0.4 \cdot c) \; P \; (0.6 \cdot b + 0.4 \cdot c)$.
2. $(0.6 \cdot a + 0.4 \cdot b) \; P \; (0.4 \cdot a + 0.6 \cdot b)$.

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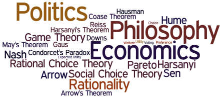
Neither of these preferences can be inferred if all you know is that the decision maker's preferences over lotteries satisfies transitivity and completeness.

A **rational** preference over lotteries involves more than the assumption that the decision maker's preferences are transitive and complete:

1. Independence axiom
2. Continuity axiom

Independence



For all $L_1, L_2, L_3 \in \mathcal{L}$ and $0 < p \leq 1$,

$$L_1 \; P \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3).$$

$$L_1 \; I \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3).$$

Independence

For all $L, L', L'' \in \mathcal{L}$ and $0 < p \leq 1$,

$$L \; P \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; P \; (p \cdot L' + (1-p) \cdot L'').$$

$$L \; I \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; I \; (p \cdot L' + (1-p) \cdot L'').$$

Independence

For all $\color{blue}{\blacksquare}$, $\color{red}{\blacksquare}$, $\color{green}{\blacksquare} \in \mathcal{L}$ and $0 < p \leq 1$,

$$\color{blue}{\blacksquare}\ \color{black}{P}\ \color{red}{\blacksquare} \quad \color{black}{\text{if, and only if,}} \quad (p \cdot \color{blue}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{)\ P\ (p \cdot} \color{red}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{).}$$

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Suppose that a decision maker has the following preference:

$$(1 \cdot \$2000) \quad P \quad (0.6 \cdot \$3000 + 0.4 \cdot \$0)$$

Assuming that the decision maker satisfies the Independence Axiom, what is the decision maker's preference between the following two lotteries?

- $L_1 = 0.5 \cdot \$2000 + 0.5 \cdot \$0$
- $L_2 = 0.3 \cdot \$3000 + 0.7 \cdot \$0$

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We can show that the decision maker must strictly prefer L_1 to L_2

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iff (Independence)

$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

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$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

iff (Simplifying lotteries)

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad ((0.5 \times 0.6) \cdot \$3000 + (0.5 \times 0.4 + 0.5) \cdot \$0)$$

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad (0.3 \cdot \$3000 + 0.7 \cdot \$0)$$