PHPE 400
Individual and Group Decision Making

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Comparing Lotteries

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Given a rational preference on X, how should the decision maker compare lotteries?

What additional properties should a *rational preference* (P, I) on $\mathcal{L}(X)$ satisfy?

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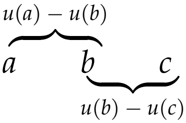

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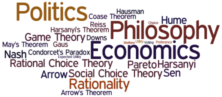

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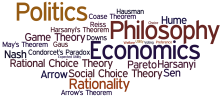
> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592kg) is twice as much as 5lb (= 2.26796kg).

Measuring Utility

L. Narens and B. Skyrms (2020). *The Pursuit of Happiness: Philosophical and Psychological Foundations of Utility*. Oxford University Press.

I. Moscati (2018). *Measuring Utility From the Marginal Revolution to Behavioral Economics*. Oxford University Press.

Expected Value of a Lottery



Suppose that the outcomes of a lottery are monetary values:

$$L = p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n$$

where each x_i is an amount of money.

The **expected value** of L is:

$$\begin{aligned} EV(p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n) &= p_1 \times x_1 + \cdots + p_n \times x_n \\ &= \sum_{i=1}^{n} p_i \times x_i \end{aligned}$$

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E.g., if $L = 0.55 \cdot \$100 + 0.25 \cdot \$50 + 0.2 \cdot \$0$, then

$$EV(L) = 0.55 \times 100 + 0.25 \times 50 + 0.2 \times 0 = 67.5$$

You are given a choice between two lotteries L_1 and L_2. The outcome of the lotteries is determined by flipping a fair coin. The payoff for the two lotteries are given in the following table:

	Heads	Tails
L_1	\$1M	\$1M
L_2	\$3M	\$0

Which of the two lotteries would you choose?

1. L_1
2. L_2
3. I am indifferent between the two lotteries

Problems with using monetary payoffs

- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.). Also, we care about more things than money.

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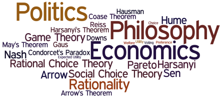

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- The St. Petersburg Paradox: Consider the following wager: I will flip a fair coin until it comes up heads; if the first time it comes up heads is the n^{th} toss, then I will pay you 2^n. What's the most you'd be willing to pay for this wager? What is its expected monetary value?

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- Risk-aversion: Is it irrational to prefer a sure-thing \$x to a wager whose expected payout is \$x?

Solution: Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \to \mathbb{R}$ is a utility function on X.

The **expected utility** of a lottery L with respect to u is defined as follows:

$$
\begin{aligned}
EU(p_1 \cdot x_1 + \cdots + p_n \cdot x_n, u) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\
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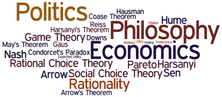

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Ordinal Utility and Expected Utility

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Consider the lotteries $L_1 = 0.5 \cdot a + 0.5 \cdot c$ and $L_2 = 1 \cdot b$

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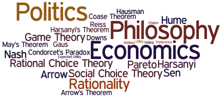
Problem: u_1, u_2, and u_3 each represent the decision maker's preferences, but rank L_1 and L_2 differently according to the expected utility.

Linear Transformations

This problem does not arise for utility functions that are **linear transformations** of each other.

Linear Transformations



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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

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u_2	0.75	0.5	0	linear transformation

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	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

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  - if $EU(L, u) = EU(L', u)$ then $EU(L, u') = EU(L', u')$
- For all $a, b, c, d \in X$,
  - if $u(a) - u(b) < u(c) - u(d)$, then $u'(a) - u'(b) < u'(c) - u'(d)$;
  - if $u(a) - u(b) > u(c) - u(d)$, then $u'(a) - u'(b) > u'(c) - u'(d)$; and
  - if $u(a) - u(b) = u(c) - u(d)$, then $u'(a) - u'(b) = u'(c) - u'(d)$.

Taking Stock

- Expected value and expected utility (with respect to some utility function) are often used to compare lotteries.
- Comparing lotteries by their expected values may result in a different ranking than comparing lotteries by their expected utility with respect to some utility function.
- To calculate the expected utility of a lottery we need the decision maker's utility function on the outcomes.