PHPE 400
Individual and Group Decision Making

Eric Pacuit
University of Maryland
pacuit.org

Lotteries

Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of outcomes.

A **lottery** over X is a tuple is a function that assigns to each outcome o the probability that o obtains. That is, it is a function $p : X \to [0, 1]$ such that

$$p(x_1) + p(x_2) + \cdots + p(x_n) = \sum_{x \in X} p(x) = 1$$

Lotteries: Example

Suppose that $X = \{a, b, c\}$. The lottery $p : X \to [0, 1]$ that assigns:

- a probability of 0.6 to a (denoted $p(a) = 0.6$),
- a probability of 0.1 to b (denoted $p(b) = 0.1$), and
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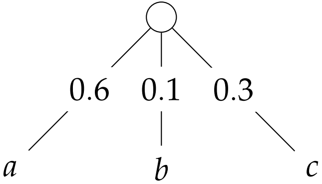

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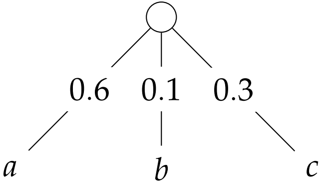

0.6	0.1	0.3
a	b	c

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$$0.25 \cdot L_1 + 0.75 \cdot L_2 \quad = \quad 0.25 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.75 \cdot (0.4 \cdot b + 0.6 \cdot c)$$

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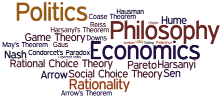

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$$\begin{aligned} 0.25 \cdot L_1 + 0.75 \cdot L_2 &= 0.25 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.75 \cdot (0.4 \cdot b + 0.6 \cdot c) \\ &= (0.25 \times 0.5) \cdot a + (0.25 \times 0.5 + 0.75 \times 0.4) \cdot b \\ &\quad +(0.75 \times 0.6) \cdot c \end{aligned}$$

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&= (0.25 \times 0.5) \cdot a + (0.25 \times 0.5 + 0.75 \times 0.4) \cdot b \\
&\quad + (0.75 \times 0.6) \cdot c \\
&= 0.125 \cdot a + 0.425 \cdot b + 0.45 \cdot c
\end{aligned}$$

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Suppose that X is a set of outcomes and $\mathcal{L}(X)$ is the set of all lotteries over X.

Given a rational preference on X, how should the decision maker compare lotteries?

What additional properties should a *rational preference* (P, I) on $\mathcal{L}(X)$ satisfy?

Comparing Lotteries

Suppose that $X = \{a, b, c\}$ and the decision maker has the strict preference

$$a \; P \; b \; P \; c$$

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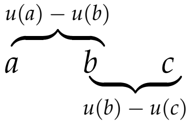

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Ordinal Utility: Qualitative comparisons of objects allowed, no information about differences or ratios.

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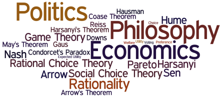

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Cardinal Utility:

> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
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> E.g., the difference between 75°F and 70°F is the same as the difference between 30°F and 25°F However, 70°F (= 21.11°C) is **not** twice as hot as 35°F (= 1.67°C).

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Ratio scale: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592kg) is twice as much as 5lb (= 2.26796kg).

Measuring Utility

L. Narens and B. Skyrms (2020). *The Pursuit of Happiness: Philosophical and Psychological Foundations of Utility*. Oxford University Press.

I. Moscati (2018). *Measuring Utility From the Marginal Revolution to Behavioral Economics*. Oxford University Press.

Expected utility

Expected Value of a Lottery

Suppose that the outcomes of a lottery are monetary values:

$$L = p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n$$

where each x_i is an amount of money.

The **expected value** of L is:

$$\begin{aligned} EV(p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n) &= p_1 \times x_1 + \cdots + p_n \times x_n \\ &= \sum_{i=1}^{n} p_i \times x_i \end{aligned}$$

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E.g., if $L = 0.55 \cdot \$100 + 0.25 \cdot \$50 + 0.2 \cdot \$0$, then

$$EV(L) = 0.55 \times 100 + 0.25 \times 50 + 0.2 \times 0 = 67.5$$

You are given a choice between two lotteries L_1 and L_2. The outcome of the lotteries is determined by flipping a fair coin. The payoff for the two lotteries are given in the following table:

	Heads	Tails
L_1	\$1M	\$1M
L_2	\$3M	\$0

Which of the two lotteries would you choose?

1. L_1
2. L_2
3. I am indifferent between the two lotteries

Solution: Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \to \mathbb{R}$ is a utility function on X.

The **expected utility** of a lottery L with respect to u is defined as follows:

$$\begin{aligned} EU(p_1 \cdot x_1 + \cdots + p_n \cdot x_n, u) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\ &= \sum_{i=1}^{n} p_i \times u(x_i) \end{aligned}$$

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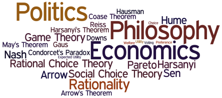

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Problem: u_1, u_2, and u_3 each represent the decision maker's preferences, but rank L_1 and L_2 differently according to the expected utility.