PHPE 400
Individual and Group Decision Making

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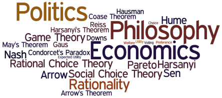
Rational Preferences



An individual's preferences are **rational** when they satisfy two additional constraints:

1. transitivity
2. completeness

Transitivity

There are two ways that a decision maker's strict preference P on X may fail transitivity:

✗ There is a *cycle* in the decision maker's preferences: There are $x, y, z \in X$ such that $x \; P \; y$, $y \; P \; z$, and $z \; P \; x$.

$\Rightarrow$ Money-pump argument, rankings, $\ldots$

1. The decision maker lacks a strict preference: There are $x, y, z \in X$ such that $x \; P \; y$ and $y \; P \; z$, but $x \; N \; z$ (i.e., x and z are incomparable).

Completeness

Completeness: For all $x, y \in X$, exactly one the following is true:

- $x \; P \; y$ (x is strictly preferred to y),
- $y \; P \; x$ (y is strictly preferred to x), or
- $x \; I \; y$ (x and y are indifferent).

I.e., for all $x, y \in X$, not-$x \; N \; y$.

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[O]f all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others, it is inaccurate as a description of real life; but unlike them we find it hard to accept even from the normative viewpoint.

(Aumann, 1962)

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  $\Rightarrow$ Completeness

Rational Preferences

A pair (P, I) is a **rational preference** on X provided that $P \subseteq X \times X$ and $I \subseteq X \times X$, such that

- P is asymmetric and transitive. That is, P is a **strict weak order**.
- I is reflexive, symmetric, and transitive. That is, P is an **equivalence relation**.
- Completeness: For all $x, y \in X$, exactly one of $x\ P\ y$, $y\ P\ x$ or $x\ I\ y$ is true.

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Note: We only need to define a strict preference relation P since I can be inferred assuming Completeness (e.g., if not-$x\ P\ y$ and not-$y\ P\ x$, then the decision maker must be indifferent between x and y).

Rational Choice

Suppose that X is set and $A \subseteq X$, and that (P, I) is a rational preference on X representing a decision maker's preferences.

x is a **maximal element** of A with respect to P when there is no element of A that is *strictly preferred* to x (i.e., there is no $y \in A$ such that $y\ P\ x$).

$x \in A$ is a **rational choice** for the decision maker if x is a **maximal element** of A with respect to P.

- Utility functions
    - Ordinal utility functions
    - Cardinal utility functions
- Lotteries

Utility Function

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

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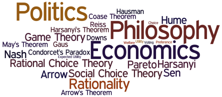
A rational preference (P, I) on X, is **represented** by a utility function $u : X \to \mathbb{R}$ if, and only if,

1. for all $x, y \in X$, $x \; P \; y$ when $u(x) > u(y)$
2. for all $x, y \in X$, $x \; I \; y$ when $u(x) = u(y)$

Ordinal Utility Theory

Fact. Suppose that X is finite and (P, I) is a rational preference on X. Then, there is a utility function $u : X \to \mathbb{R}$ that represents (P, I).

Ordinal Utility Theory



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Utility is *defined* in terms of the decision maker's preference, **so it is an error to say that the decision maker prefers x to y *because* she assigns a higher utility to x than to y.**

Important

All three of the utility functions represent the preference $x \; P \; y \; P \; z$

	x	y	z
u_1	3	2	1
u_2	10	5	0
u_3	1000	999	1

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$x \; P \; y \; P \; z$ is represented by both u_1 and u_3, so one cannot say that y is “closer” to x than to z:

- $u_1(x) - u_1(y) = 1 = u_1(y) - u_1(z)$
- $u_3(x) - u_3(y) = 1 < 998 = u_3(y) - u_3(z)$

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Lotteries

Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of outcomes.

A **lottery** over X is a tuple is a function that assigns to each outcome o the probability that o obtains. That is, it is a function $p : X \to [0,1]$ such that

$$p(x_1) + p(x_2) + \cdots + p(x_n) = \sum_{x \in X} p(x) = 1$$

Lotteries: Example

Suppose that $X = \{a, b, c\}$. The lottery $p : X \rightarrow [0, 1]$ that assigns:

- a probability of 0.6 to a (denoted $p(a) = 0.6$),
- a probability of 0.1 to b (denoted $p(b) = 0.1$), and
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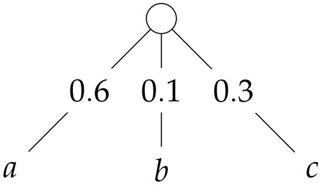

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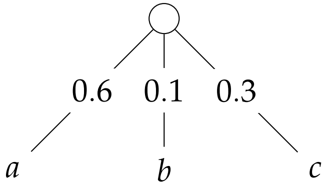

0.6	0.1	0.3
a	b	c

Important Point about Lotteries

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Let $L_1 = 0.5 \cdot a + 0.5 \cdot b$ and $L_2 = 0.4 \cdot b + 0.6 \cdot c$.

Then the lottery $0.25 \cdot L_1 + 0.75 \cdot L_2$ can be simplified as follows:

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$$0.25 \cdot L_1 + 0.75 \cdot L_2 \quad = \quad 0.25 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.75 \cdot (0.4 \cdot b + 0.6 \cdot c)$$

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$$\begin{aligned} 0.25 \cdot L_1 + 0.75 \cdot L_2 &= 0.25 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.75 \cdot (0.4 \cdot b + 0.6 \cdot c) \\ &= (0.25 \times 0.5) \cdot a + (0.25 \times 0.5 + 0.75 \times 0.4) \cdot b \\ &\quad +(0.75 \times 0.6) \cdot c \end{aligned}$$

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&= (0.25 \times 0.5) \cdot a + (0.25 \times 0.5 + 0.75 \times 0.4) \cdot b \\
&\quad + (0.75 \times 0.6) \cdot c \\
&= 0.125 \cdot a + 0.425 \cdot b + 0.45 \cdot c
\end{aligned}$$

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Suppose that X is a set of outcomes and $\mathcal{L}(X)$ is the set of all lotteries over X.

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Given a rational preference on X, how should the decision maker compare lotteries?

What additional properties should a *rational preference* (P, I) on $\mathcal{L}(X)$ satisfy?

Comparing Lotteries

Suppose that $X = \{a, b, c\}$ and the decision maker has the strict preference

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$$\overbrace{a \qquad\quad b}^{u(a)-u(b)}\!\!\!\underbrace{\qquad\quad c}_{u(b)-u(c)}$$

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> **Interval scale**: Quantitative comparisons of objects, accurately reflects differences between objects.
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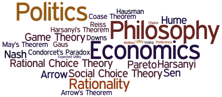

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> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592*kg*) is twice as much as 5lb (= 2.26796*kg*).

Measuring Utility

L. Narens and B. Skyrms (2020). *The Pursuit of Happiness: Philosophical and Psychological Foundations of Utility*. Oxford University Press.

I. Moscati (2018). *Measuring Utility From the Marginal Revolution to Behavioral Economics*. Oxford University Press.

Expected utility

Expected Value of a Lottery

Suppose that the outcomes of a lottery are monetary values:

$$L = p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n$$

where each x_i is an amount of money.

The **expected value** of L is:

$$\begin{aligned} EV(p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n) &= p_1 \times x_1 + \cdots + p_n \times x_n \\ &= \sum_{i=1}^{n} p_i \times x_i \end{aligned}$$

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E.g., if $L = 0.55 \cdot \$100 + 0.25 \cdot \$50 + 0.2 \cdot \$0$, then

$$EV(L) = 0.55 \times 100 + 0.25 \times 50 + 0.2 \times 0 = 67.5$$

You are given a choice between two lotteries L_1 and L_2. The outcome of the lotteries is determined by flipping a fair coin. The payoff for the two lotteries are given in the following table:

	Heads	Tails
L_1	\$1M	\$1M
L_2	\$3M	\$0

Which of the two lotteries would you choose?

1. L_1
2. L_2
3. I am indifferent between the two lotteries

Problems with using monetary payoffs

- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.). Also, we care about more things than money.

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- The St. Petersburg Paradox: Consider the following wager: I will flip a fair coin until it comes up heads; if the first time it comes up heads is the n^{th} toss, then I will pay you 2^n. What's the most you'd be willing to pay for this wager? What is its expected monetary value?

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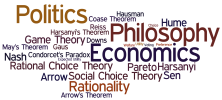

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- Risk-aversion: Is it irrational to prefer a sure-thing \$x to a wager whose expected payout is \$x?

Solution: Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \to \mathbb{R}$ is a utility function on X.

The **expected utility** of a lottery L with respect to u is defined as follows:

$$\begin{aligned} EU(p_1 \cdot x_1 + \cdots + p_n \cdot x_n, u) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\ &= \textstyle\sum_{i=1}^{n} p_i \times u(x_i) \end{aligned}$$

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Ordinal Utility and Expected Utility

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Consider the lotteries $L_1 = 0.5 \cdot a + 0.5 \cdot c$ and $L_2 = 1 \cdot b$

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u_3	4	3	1	$u_3(a) > u_3(b) > u_3(c)$	$EU(L_1, u_3) < EU(L_2, u_3)$

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u_1	4	1.5	1	$u_1(a) > u_1(b) > u_1(c)$	$EU(L_1, u_1) > EU(L_2, u_1)$
u_2	4	2.5	1	$u_2(a) > u_2(b) > u_2(c)$	$EU(L_1, u_2) = EU(L_2, u_2)$
u_3	4	3	1	$u_3(a) > u_3(b) > u_3(c)$	$EU(L_1, u_3) < EU(L_2, u_3)$

Problem: u_1, u_2, and u_3 each represent the decision maker's preferences, but rank L_1 and L_2 differently according to the expected utility.

Linear Transformations

This problem does not arise for utility functions that are **linear transformations** of each other.

Linear Transformations

Suppose that $u : X \to \mathbb{R}$ is a utility function. We say that $u' : X \to \mathbb{R}$ is a **linear transformation of** u provided that there are numbers $\alpha > 0$ and β such that for all $x \in X$: (also called **positive affine transformation**)

$$u'(x) = \alpha \times u(x) + \beta$$

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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

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u_1	32	22	2	linear transformation
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	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

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Suppose that $u : X \to \mathbb{R}$ is a utility function on X and $u' : X \to \mathbb{R}$ is a linear transformation of u.

- For all lotteries L and L',
  - if $EU(L, u) > EU(L', u)$ then $EU(L, u') > EU(L', u')$
  - if $EU(L, u) < EU(L', u)$ then $EU(L, u') < EU(L', u')$
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  - if $EU(L, u) = EU(L', u)$ then $EU(L, u') = EU(L', u')$
- For all $a, b, c, d \in X$,
  - if $u(a) - u(b) < u(c) - u(d)$, then $u'(a) - u'(b) < u'(c) - u'(d)$;
  - if $u(a) - u(b) > u(c) - u(d)$, then $u'(a) - u'(b) > u'(c) - u'(d)$; and
  - if $u(a) - u(b) = u(c) - u(d)$, then $u'(a) - u'(b) = u'(c) - u'(d)$.

Taking Stock

- Expected value and expected utility (with respect to some utility function) are often used to compare lotteries.

- Comparing lotteries by their expected values may result in a different ranking than comparing lotteries by their expected utility with respect to some utility function.

- To calculate the expected utility of a lottery we need the decision maker's utility function on the outcomes.

Comparing Lotteries

Suppose that X is a set of outcomes and $\mathcal{L}(X)$ is the set of all lotteries over X.

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Answer: Compare lotteries according to their expected utility with respect to a cardinal utility function on X representing the decision maker's preferences.

What additional properties should a *rational preference* (P, I) on $\mathcal{L}(X)$ satisfy?

Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b) and c is another item.

How *should* the decision maker rank the lotteries

$$L_1 = 0.6 \cdot a + 0.4 \cdot c \quad \text{and} \quad L_2 = 0.6 \cdot b + 0.4 \cdot c?$$

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1. $L_1 \; P \; L_2$: The decision maker should strictly prefer L_1 to L_2.
2. $L_2 \; P \; L_1$: The decision maker should strictly prefer L_2 to L_1.
3. $L_1 \; I \; L_2$: The decision maker should be indifferent between L_1 and L_2.
4. There is not enough information to answer this question.

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3. $L_1 \; I \; L_2$: The decision maker should be indifferent between L_1 and L_2.
4. There is not enough information to answer this question.

Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b) and c is another item.

Then, a *rational* decision maker will have the following preferences:

1. $(0.6 \cdot a + 0.4 \cdot c) \; P \; (0.6 \cdot b + 0.4 \cdot c)$.
2. $(0.6 \cdot a + 0.4 \cdot b) \; P \; (0.4 \cdot a + 0.6 \cdot b)$.

Suppose that the decision maker is rational and has the preference $a \, P \, b$ (the decision maker strictly prefers a to b) and c is another item.

Then, a *rational* decision maker will have the following preferences:

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2. $(0.6 \cdot a + 0.4 \cdot b) \; P \; (0.4 \cdot a + 0.6 \cdot b)$.

Neither of these preferences can be inferred if all you know is that the decision maker's preferences over lotteries satisfies transitivity and completeness.

A **rational** preference over lotteries involves more than the assumption that the decision maker's preferences are transitive and complete:

1. Independence axiom
2. Continuity axiom

Independence

For all $L_1, L_2, L_3 \in \mathcal{L}$ and $0 < p \leq 1$,

$$L_1 \; P \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3).$$

$$L_1 \; I \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3).$$

Independence

For all $L, L', L'' \in \mathcal{L}$ and $0 < p \leq 1$,

$$L \; P \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; P \; (p \cdot L' + (1-p) \cdot L'').$$

$$L \; I \; L' \quad \text{if, and only if,} \quad (p \cdot L + (1-p) \cdot L'') \; I \; (p \cdot L' + (1-p) \cdot L'').$$

Independence

For all $\color{blue}{\blacksquare}$, $\color{red}{\blacksquare}$, $\color{green}{\blacksquare} \in \mathcal{L}$ and $0 < p \leq 1$,

$$\color{blue}{\blacksquare}\ \color{black}{P}\ \color{red}{\blacksquare} \quad \color{black}{\text{if, and only if,}} \quad (p \cdot \color{blue}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{)\ P\ (p \cdot} \color{red}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{).}$$

$$\color{blue}{\blacksquare}\ \color{black}{I}\ \color{red}{\blacksquare} \quad \color{black}{\text{if, and only if,}} \quad (p \cdot \color{blue}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{)\ I\ (p \cdot} \color{red}{\blacksquare} \color{black}{+ (1-p) \cdot} \color{green}{\blacksquare}\color{black}{).}$$

Suppose that a decision maker has the following preference:

$$(1 \cdot \$2000) \quad P \quad (0.6 \cdot \$3000 + 0.4 \cdot \$0)$$

Assuming that the decision maker satisfies the Independence Axiom, what is the decision maker's preference between the following two lotteries?

- $L_1 = 0.5 \cdot \$2000 + 0.5 \cdot \$0$
- $L_2 = 0.3 \cdot \$3000 + 0.7 \cdot \$0$

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- $L_2 = 0.3 \cdot \$3000 + 0.7 \cdot \$0$

We can show that the decision maker must strictly prefer L_1 to L_2

$$(1 \cdot \$2000) \quad P \quad (0.6 \cdot \$3000 + 0.4 \cdot \$0)$$

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iff (Independence)

$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

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iff (Independence)

$$(0.5 \cdot (1 \cdot \$2000) + 0.5 \cdot \$0) \quad P \quad (0.5 \cdot (0.6 \cdot \$3000 + 0.4 \cdot \$0) + 0.5 \cdot \$0)$$

iff (Simplifying lotteries)

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad ((0.5 \times 0.6) \cdot \$3000 + (0.5 \times 0.4 + 0.5) \cdot \$0)$$

$$(0.5 \cdot \$2000 + 0.5 \cdot \$0) \quad P \quad (0.3 \cdot \$3000 + 0.7 \cdot \$0)$$

Independence

For all $L_1, L_2, L_3 \in \mathcal{L}$ and $0 < p \leq 1$,

$$L_1 \; P \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3).$$

$$L_1 \; I \; L_2 \quad \text{if, and only if,} \quad (p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3).$$

Independence

For all ${\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{green}\blacksquare} \in \mathcal{L}$ and $0 < p \leq 1$,

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Example

Consider the set of all lotteries over $X = \{a, b\}$.

Suppose that Ann prefers lotteries that are closer to 50-50. For example,

- $(0.5 \cdot a + 0.5 \cdot b) \;\; P \;\; (0.25 \cdot a + 0.75 \cdot b)$
- $(0.75 \cdot a + 0.25 \cdot b) \;\; I \;\; (0.25 \cdot a + 0.75 \cdot b)$
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- $(0.25 \cdot a + 0.75 \cdot b) \;\; P \;\; (1 \cdot a)$

We can view Ann as assigning a value to any lottery as follows:

$$\text{A lottery } r \cdot a + (1 - r) \cdot b \text{ is valued at } -|r - \tfrac{1}{2}|.$$

Then, Ann ranks lotteries by assigning a value to the lotteries and ranking them according to the values.

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- Do Ann’s preferences satisfy the Independence Axiom? No.

Ann's preferences violates the Independence Axiom since she has the following preferences:

- $(0.5 \cdot a + 0.5 \cdot b) \;\; P \;\; (1 \cdot a)$
- It is **not** the case that

$$(0.5 \cdot \;(0.5 \cdot a + 0.5 \cdot b)\; + 0.5 \cdot \; b \;)\; P \;(0.5 \cdot \;(1 \cdot a)\; + 0.5 \cdot \; b \;).$$

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$$(0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b) \; P \; (0.5 \cdot (1 \cdot a) + 0.5 \cdot b).$$

This is because

$$\begin{aligned} (0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b) &= (0.25 \cdot a + 0.75 \cdot b) \\ (0.5 \cdot (1 \cdot a) + 0.5 \cdot b) &= (0.5 \cdot a + 0.5 \cdot b) \end{aligned}$$

And so, $(0.5 \cdot a + 0.5 \cdot b) \; P \; (0.25 \cdot a + 0.75 \cdot b)$ since the value of $(0.5 \cdot a + 0.5 \cdot b)$ is 0 and $(0.25 \cdot a + 0.75 \cdot b)$ is -0.25 and $-0.25 < 0$. Hence, it is not the case that $(0.5 \cdot (0.5 \cdot a + 0.5 \cdot b) + 0.5 \cdot b) \; P \; (0.5 \cdot (1 \cdot a) + 0.5 \cdot b)$.

Violating Independence

A decision maker **does not** satisfy the Independence Axiom when there are lotteries L_1, L_2, L_3 and a number p such that $0 < p \leq 1$ such that at least one of the following is true:

1. $L_1 \; P \; L_2$, but it is not the case that $(p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3)$;
2. $(p \cdot L_1 + (1-p) \cdot L_3) \; P \; (p \cdot L_2 + (1-p) \cdot L_3)$, but it is not the case that $L_1 \; P \; L_2$;
3. $L_1 \; I \; L_2$, but it is not the case that $(p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3)$; or
4. $(p \cdot L_1 + (1-p) \cdot L_3) \; I \; (p \cdot L_2 + (1-p) \cdot L_3)$, but it is not the case that $L_1 \; I \; L_2$;

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- For a *cardinal* utility function (measured on an interval scale) that represents a decision maker's preferences over outcomes, the decision maker compares lotteries using **expected utility** based on this utility function.

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- For a *cardinal* utility function (measured on an interval scale) that represents a decision maker's preferences over outcomes, the decision maker compares lotteries using **expected utility** based on this utility function.
  - The utility function representing the decision maker's preferences is unique up to *linear transformations*.

- **Rational preferences** over lotteries are characterized by satisfying Transitivity, Completeness, and the Independence Axiom.
  - If a decision maker violates the Independence Axiom, then no utility function can be used to rank lotteries using expected utility.