PHPE 400
Individual and Group Decision Making

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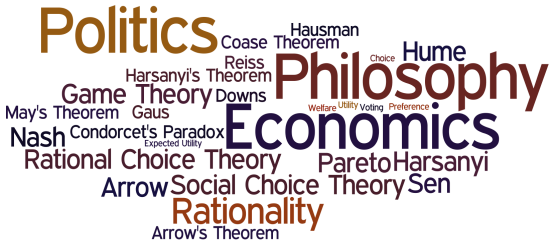

Utility Functions

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A preference ordering is **represented** by a utility function u if, and only if, (i) x is strictly preferred to y when $u(x) > u(y)$ and (ii) x and y are indifferent when $u(x) = u(y)$.

L. Narens and B. Skyrms . *The Pursuit of Happiness Philosophical and Psychological Foundations of Utility*. Oxford University Press, 2020.

Let X and V be nonempty sets with $|X| \geq 3$ and V finite.

Let $\mathcal{U}(X)$ be the set of all functions $u : X \to \mathbb{R}$

A **profile** is a function $\mathbf{U} : V \to \mathcal{U}(X)$, write $\mathbf{U}_i$ for voter i's utility function on X in profile $\mathbf{U}$.

A **Social Welfare Functional (SWFL)** is a function f mapping profiles of utilities to asymmetric relations on X. So for each profile $\mathbf{U}$, $f(\mathbf{U})$ is the social preference order on X.

Sum Utilitarian: Define f_U as follows: For all $x, y \in X$,

$$x \, f_U(\mathbf{U}) \, y \text{ if and only if } \sum_i \mathbf{U}_i(x) \geq \sum_i \mathbf{U}_i(y)$$

$\mathbf{U}$	x	y	z
v_1	3	1	8
v_2	3	2	1
v_3	1	4	1

U	x	y	z
v_1	3	1	8
v_2	3	2	1
v_3	1	4	1
Sum	7	7	10

- Sum utilitarian: z is ranked above x and y, and x and y are tied.

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This satisfies versions of Arrow's axioms, including non-dictatorship!

Arrow Axioms

Transitivity/Completeness: For all $\mathbf{U}$ in the domain of f, $f(\mathbf{U})$ is transitive/complete.

Universal Domain: the domain of f is the set of all profiles

Pareto: For all $\mathbf{U}$ in the domain of f, for all $x, y \in X$, if $\mathbf{U}_i(x) > \mathbf{U}_i(y)$ for all $i \in V$, then x is ranked strictly above y according to $f(\mathbf{U})$.

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Independence of Irrelevant Utilities: For all $\mathbf{U}$ and $\mathbf{U}'$ in the domain of f, for all $x, y \in X$, if $\mathbf{U}_i(x) = \mathbf{U}'_i(x)$ and $\mathbf{U}_i(y) = \mathbf{U}'_i(y)$ for all $i \in V$, then $x \, f(\mathbf{U}) \, y$ if and only if $x \, f(\mathbf{U}') \, y$.

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(*Social Choice and Individual Values*, p. 9).

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Why not?

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(*Social Choice and Individual Values*, pp. 10-11).

Linear Transformations

Suppose that $u : X \to \mathbb{R}$ is a utility function. We say that $u' : X \to \mathbb{R}$ is a **linear transformation of** u provided that there are numbers $\alpha > 0$ and β such that for all $x \in X$:

$$u'(x) = \alpha \times u(x) + \beta$$

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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation

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u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

According to standard understanding of utilities in rational choice (as used throughout Economics, Philosophy and Political Science), a decision maker's utility is **unique up to linear transformations**.

U	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
z
x y

U	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
x
y
z

U	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
y
x
z

Equivalent profiles from the vNM perspective

Cardinal measurability equivalence: Given two profiles $\mathbf{U}$ and $\mathbf{U}'$, let $\mathbf{U} \sim_{CM} \mathbf{U}'$ if for every $i \in V$, there are $\alpha_i, \beta_i \in \mathbb{R}$ with $\alpha_i > 0$ such that for all $x \in X$, $\mathbf{U}_i(x) = \alpha_i \times \mathbf{U}'_i(x) + \beta_i$.

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The following profiles are all cardinal measurability equivalent:

$\mathbf{U}$	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

$\mathbf{U}'$	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

$\mathbf{U}''$	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

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This suggests any SWFL should give the same output for any two such profiles:

An Social Welfare Functional f satisfies **CM-invariance** if for all $\mathbf{U}, \mathbf{U}'$, if $\mathbf{U} \sim_{CM} \mathbf{U}'$, then $f(\mathbf{U}) = f(\mathbf{U}')$.

Arrow's theorem

We can now state an update of Arrow's Impossibility Theorem developed by Amartya Sen:

Theorem. Assume X is a set of candidates with at least 3 elements and that V is finite. If f if an SWFL satisfying **Universal Domain**, **Pareto**, **CM-invariance**, **Independence of Irrelevant Utilities**, and **Rationality**, then f is a *dictatorship*: there is some $i \in V$ such that for all profiles $\mathbf{U}$ and $x, y \in X$, if $\mathbf{U}_i(x) > \mathbf{U}_i(y)$, then $x \, f(\mathbf{U}) \, y$.

What now?

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What now?

In social welfare theory a standard response has been to replace **CM-invariance** by another equivalence relation, assuming a greater degree of *interpersonal comparability of utility*.

Interpersonal Comparison of Utility

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Arrow: "...It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula. If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual. Even Bentham had his doubts on this point."

(*Social Choice and Individual Values*, p. 11).

Mary seashore P museums P camping

Sam camping P museums P seashore

- The seashore is the only alternative that Mary finds bearable, although she feels more negative about going to the mountains than to the museums.
- Each choice is fine with Sam, although he would much prefer going to the mountains.

	Mary	Sam	Total
Seashore	20		
Museums	10		
Mountains	9		

	Mary	Sam	Total
Seashore	20	86	
Museums	10	93	
Mountains	9	100	

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

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Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

For Mary, the difference between the seashore and the mountains crosses the threshold between the bearable and the intolerable. She feels that her “right to an emotionally recuperative vacation will be violated by following a utilitarian scheme.

	Mary	Sam	Total
Seashore	200	86	286
Museums	100	93	190
Mountains	90	100	190

Mary: My preferences are so intense in comparison with yours that my scale should range between 0 and 1,000, if yours range between 0 and 100.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

Sam: You think that my preferences are rather weak, but the fact is I feel things quite deeply. I have been brought up in a culture very different from yours and have been trained to avoid emotional outbursts...But I have strong feelings all the same.

	Mary	Sam	Total
Seashore	20	86	106
Museums	10	93	103
Mountains	9	100	109

Sam: I do not think that extra weight *should* be given in a utilitarian calculation to those who are capable of more intense preferences.