Utility Profiles and Social Welfare Functionals

Utility Functions

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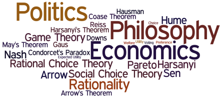

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A preference ordering is **represented** by a utility function u if, and only if, (i) x is strictly preferred to y when $u(x) > u(y)$ and (ii) x and y are indifferent when $u(x) = u(y)$.

L. Narens and B. Skyrms . *The Pursuit of Happiness Philosophical and Psychological Foundations of Utility*. Oxford University Press, 2020.

Let X and V be nonempty sets with $|X| \geq 3$ and V finite.

Let $\mathcal{U}(X)$ be the set of all functions $u : X \to \mathbb{R}$

A **profile** is a function $\mathbf{U} : V \to \mathcal{U}(X)$, write $\mathbf{U}_i$ for voter i's utility function on X in profile $\mathbf{U}$.

A **Social Welfare Functional (SWFL)** is a function f mapping profiles of utilities to asymmetric relations on X. So for each profile $\mathbf{U}$, $f(\mathbf{U})$ is the social preference order on X.

Sum Utilitarian: Define f_U as follows: For all $x, y \in X$,

$$x \, f_U(\mathbf{U}) \, y \text{ if and only if } \sum_i \mathbf{U}_i(x) \geq \sum_i \mathbf{U}_i(y)$$

$\mathbf{U}$	x	y	z
v_1	3	1	8
v_2	3	2	1
v_3	1	4	1

U	x	y	z
v_1	3	1	8
v_2	3	2	1
v_3	1	4	1
Sum	7	7	10

- Sum utilitarian: z is ranked above x and y, and x and y are tied.

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This satisfies versions of Arrow's axioms, including non-dictatorship!

Arrow Axioms

Universal Domain: the domain of f is the set of all profiles

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Independence of Irrelevant Utilities: For all $\mathbf{U}$ and $\mathbf{U}'$ in the domain of f, for all $x, y \in X$, if $\mathbf{U}_i(x) = \mathbf{U}'_i(x)$ and $\mathbf{U}_i(y) = \mathbf{U}'_i(y)$ for all $i \in V$, then $x\, f(\mathbf{U})\, y$ if and only if $x\, f(\mathbf{U}')\, y$.

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Non-Dictatorship: There is no $d \in V$ such that for all profiles $\mathbf{U}$, for all $x, y \in X$, if $\mathbf{U}_d(x) > \mathbf{U}_d(y)$, then $x \, f(\mathbf{U}) \, y$.

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(*Social Choice and Individual Values*, p. 9).

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Why not?

Arrow: "At best, it is contended that, for an individual, his **utility function is uniquely determined up to a linear transformation; we must still choose one out of the infinite family of indicators to represent the individual**, and the values of the aggregate (say a sum) are dependent on how the choice is made for each individual.

In general, there seems to be no method intrinsic to utility measurement which will make the choices compatible..."

(*Social Choice and Individual Values*, pp. 10-11).

According to standard understanding of utilities in rational choice (as used throughout Economics, Philosophy and Political Science), a decision maker's utility is **unique up to linear transformations**.

U	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
z
x y

U	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
x
y
z

U	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
y
x
z

Linear Transformations

Suppose that $u : X \to \mathbb{R}$ is a utility function. We say that $u' : X \to \mathbb{R}$ is a **linear transformation of** u provided that there are numbers $\alpha > 0$ and β such that for all $x \in X$:

$$u'(x) = \alpha \times u(x) + \beta$$

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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation

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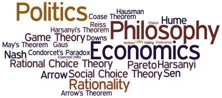
	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

Cardinal equivalence

Two profiles $\mathbf{U}$ and $\mathbf{U}'$ are **cardinally equivalent** if for every $i \in V$, the utility $\mathbf{U}'_i$ is a linear transformation of $\mathbf{U}_i$.

Cardinal equivalence



Two profiles $\mathbf{U}$ and $\mathbf{U}'$ are **cardinally equivalent** if for every $i \in V$, the utility $\mathbf{U}'_i$ is a linear transformation of $\mathbf{U}_i$.

The following profiles are all cardinally equivalent:

$\mathbf{U}$	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

$\mathbf{U}'$	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

$\mathbf{U}''$	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

Arrow claimed there is no way to tell which of two cardinally equivalent profiles is the “correct” one.

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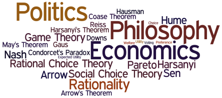
This suggests any SWFL should give the same output for any two such profiles:

An Social Welfare Functional f satisfies **Cardinal Invariance** if for all $\mathbf{U}, \mathbf{U}'$, if $\mathbf{U}$ and $\mathbf{U}'$ are cardinally equivalent, then $f(\mathbf{U}) = f(\mathbf{U}')$.



Arrow’s theorem

Amartya Sen’s Version of Arrow’s Theorem. Assume X is a set of candidates with at least 3 elements and that V is finite. There is no SWFL f satisfying **Universal Domain**, **Pareto**, **Cardinal Invariance**, **Independence of Irrelevant Utilities**, **Rationality**, and **Non-Dictatorship**.

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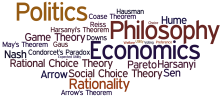

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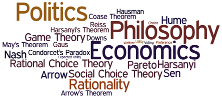
One response to Arrow’s Theorem is to drop Cardinal Invariance in favor of, for example, the sum utilitarian social welfare functional that requires **interpersonal comparisons of utility**.

Against Interpersonal Comparisons of Utility



Arrow: “...It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula.

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Arrow: "...It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula.

If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual. Even Bentham had his doubts on this point." (*Social Choice and Individual Values*, p. 11).

Example

Mary seashore P museums P camping

Sam camping P museums P seashore

- The seashore is the only alternative that Mary finds bearable, although she feels more negative about going to the mountains than to the museums.
- Each choice is fine with Sam, although he would much prefer going to the mountains.