PHPE 400
Individual and Group Decision Making

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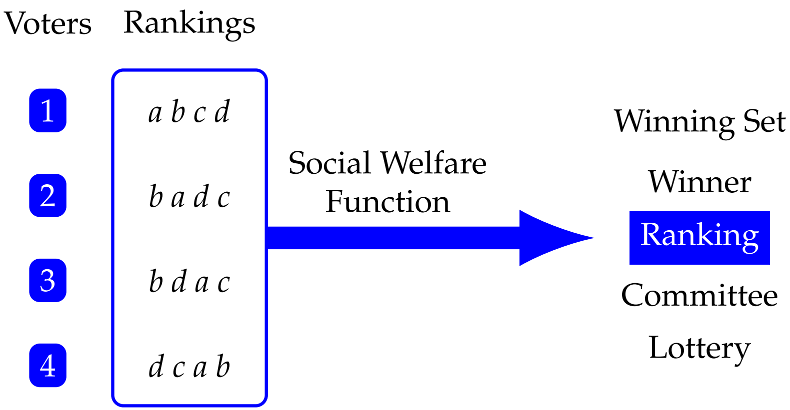
Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Social Welfare Function
Winning Set
Winner
Ranking
Committee
Lottery

Social Welfare Functions

A **Social Welfare Function** f maps an election from a set $\mathcal{D}$ of possible elections to an ordering on the set of candidates.

Comments

- $\mathcal{D}$ is called *domain* of the function f.
- Social Welfare Functions are *decisive*: every profile $\mathbf{P}$ in the domain is associated with exactly one ordering over the candidates
- For each profile $\mathbf{P}$, the ordering $f(\mathbf{P})$ is called the **social ordering** of $\mathbf{P}$ according to f.

Examples

Borda Ordering: $Borda(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that the Borda score of a is greater than or equal to the Borda score for b in the profile $\mathbf{P}$.

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Plurality Ordering: $Plurality(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that the Plurality score of a is greater than or equal to the Plurality score for b.

Majority Ordering: $Maj(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that $Margin_{\mathbf{P}}(a, b) \geq 0$

Arrow’s Axioms

Universal Domain

Voter’s are free to choose any ranking, and the voters’ choices are independent.

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"If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings."

(Arrow, p. 24)

Rationality

The social ranking is a **rational preference** on the set of candidates.

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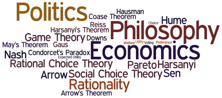
Example: Plurality and Borda always produces a complete and transitive ranking of the candidates, but the Majority ordering may output rankings that are not transitive.

Pareto/Unanimity

If each voter ranks *a* strictly above *b*, then so does the social ranking.

Pareto/Unanimity



If each voter ranks a strictly above b, then so does the social ranking.

For all profiles $\mathbf{P}$ in the domain of f: If $a\ \mathbf{P}_i\ b$ for each $i \in V$ then a is strictly preferred to b according to $f(\mathbf{P})$

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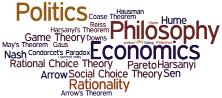
For example, Plurality violates Pareto, but Borda and the Majority Ordering both satisfy Pareto.

Voting Splitting



40	35	25
t	r	k
k	k	t
r	t	r

According to Plurality, t wins and k loses...
even though a majority of voters prefer k to t.

r is a spoiler: r splits the vote of all voters rankings k above t.

Voting Splitting

40	35	25		40	35	25
t	r	k		t	k	k
k	k	t		k	t	t
r	t	r		r	r	r

Independence of Irrelevant Alternatives: If k wins and t loses in the profile on the right, then the same should happen in the profile on the left

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The social ranking (higher, lower, or indifferent) of two alternatives a and b depends only the relative rankings of a and b for each voter.

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For all profiles $\mathbf{P}$ and $\mathbf{P}'$:

$$\text{If } \mathbf{P}_{i\{a,b\}} = \mathbf{P}'_{i\{a,b\}} \text{ for all } i \in V, \text{ then } f(\mathbf{P})_{\{a,b\}} = f(\mathbf{P}')_{\{a,b\}}.$$

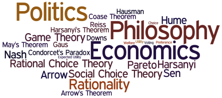
where $P_{\{x,y\}}$ is the ranking on x and y defined as follows:

$$P_{\{x,y\}} = P \cap \{x, y\} \times \{x, y\}$$

Independence of Irrelevant Alternatives

(IIA): For all profiles $\mathbf{P}, \mathbf{P}'$ and $x, y \in X$,
if $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then $f(\mathbf{P})_{\{x,y\}} = f(\mathbf{P}')_{\{x,y\}}$.

Independence of Irrelevant Alternatives



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(IIA): For all profiles $\mathbf{P}$ and all $x, y \in X$,
if $\mathbf{P}'$ is a profile in the domain of f such that $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then

- If x defeats y according to f in $\mathbf{P}$, then x defeats y according to f in $\mathbf{P}'$
- If x does not defeat y according to f in $\mathbf{P}$, then x does not defeat y according to f in $\mathbf{P}'$

Borda violates IIA, Example 1

P:

45	55	$f_{borda}(\mathbf{P})$
a	b	a
c	a	b
b	c	c

P′:

45	55	$f_{borda}(\mathbf{P}')$
a	b	b
b	a	a
c	c	c

Borda violates IIA, Example 1

$\mathbf{P}$:

45	55	$f_{borda}(\mathbf{P})$
a	b	a
c	a	b
b	c	c

$\mathbf{P}'$:

45	55	$f_{borda}(\mathbf{P}')$
a	b	b
b	a	a
c	c	c

$\mathbf{P}_{|\{a,b\}} = \mathbf{P}'_{|\{a,b\}}$, but

a beats b in $\mathbf{P}$ according to Borda, and b beats a in $\mathbf{P}'$ according to Borda.

Borda violates IIA, Example 2

P:

1	1	$f_{borda}(\mathbf{P})$
a	c	$a\ b\ c$
b	b	d
c	a	
d	d	

P′:

1	1	$f_{borda}(\mathbf{P}')$
a	c	$a\ b$
b	b	c
d	a	d
c	d	

Borda violates IIA, Example 2

	1	1
	a	c
P:	b	b
	c	a
	d	d

$f_{borda}(\mathbf{P})$
a b c
d

	1	1
	a	c
P′:	b	b
	d	a
	c	d

$f_{borda}(\mathbf{P}')$
a b
c
d

$\mathbf{P}_{|\{b,c\}} = \mathbf{P}'_{|\{b,c\}}$, but
b and c are tied in **P** according to Borda,
and b is ranked above c in **P′** accodring to Borda.

Dictatorship

A voter $d \in V$ is a **dictator** for f if society strictly prefers a over b according to f *whenever* d strictly prefers a over b.

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A voter $d \in V$ is a **dictator** for f if society strictly prefers a over b according to f *whenever* d strictly prefers a over b.

There is a $d \in V$ such that for each profile $\mathbf{P}$, if $a \ \mathbf{P}_d \ b$
then a is strictly preferred to b according to $f(\mathbf{P})$

Non-Dictatorship: There is no voter that is a dictator for f.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies Universal Domain, Rationality, Pareto, Independence of Irrelevant Alternatives (IIA) is a Dictatorship.

- Alternative statement of the theorem: Suppose that there are at least three candidates and finitely many voters. There is no social welfare function that satisfies Universal Domain, Rationality, Pareto, Independence of Irrelevant Alternatives (IIA), and Non-Dictatorship.