PHPE 400
Individual and Group Decision Making

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	Plurality	Borda	Ranked Choice	Coombs	Cope-land	Mini-max	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	—	—	—	—	✓	✓	✓
Condorcet Loser	—	✓	✓	✓	✓	—	✓
Monotonicity	✓	✓	—	—	✓	✓	✓

Positive Involvement

Positive Involvement: Suppose that C is a set of voters such had the voters in C stayed home (i.e., not voted), candidate a would have won and everyone in C ranks a first. Then, a should win in the elections with the voters from C.

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People are often shocked to learn that some standard voting methods violate Positive Involvement.

Coombs violates Positive Involvement

2	2	1	1	2	1	1
c	b	d	d	c	a	b
a	a	c	a	b	d	d
b	c	b	c	d	b	a
d	d	a	b	a	c	c

Coombs winner: $\{b\}$

(the order of elimination is d, c)

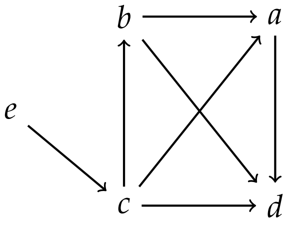
2	2	1	1	2	1	1	1
c	b	d	d	c	a	b	b
a	a	c	a	b	d	d	d
b	c	b	c	d	b	a	c
d	d	a	b	a	c	c	a

Coombs winner: $\{c\}$

(a and d are tied for the most last place votes)

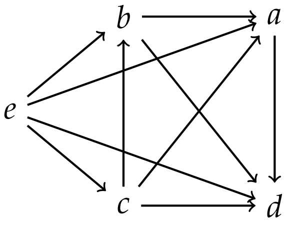
Copeland violates Positive Involvement

2	1	1
e	c	a
c	b	d
b	a	b
a	d	e
d	e	c



Copeland winners: $\{c\}$

2	1	1	1
e	c	a	c
c	b	d	e
b	a	b	d
a	d	e	c
d	e	c	a

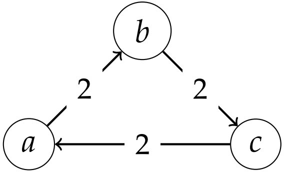


Copeland winners: $\{e\}$

Multiple-Districts Paradox

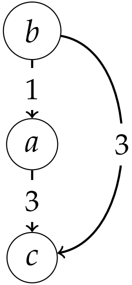
Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

Multiple-Districts Paradox



2	2	2
a	b	c
b	c	a
c	a	b

1	2
a	b
b	a
c	c



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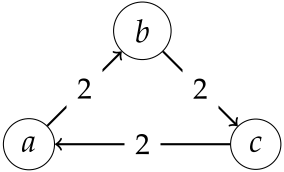
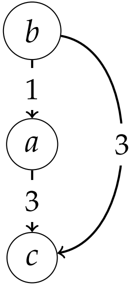
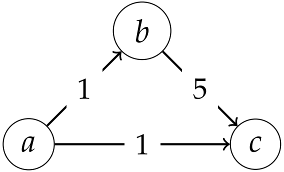


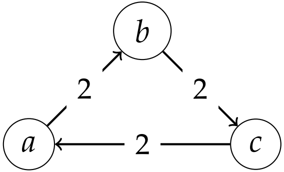
2	2	2
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

1	2
a	*b*
b	*a*
c	*c*

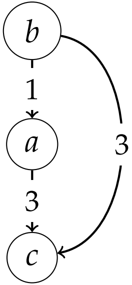


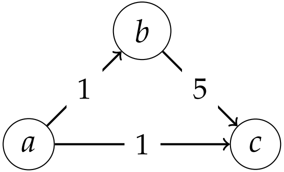


Multiple-Districts Paradox



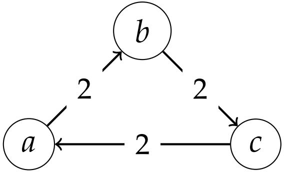
2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c



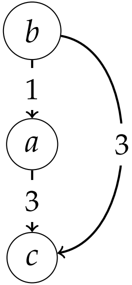


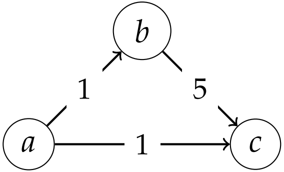
- $\{a, b, c\}$ are the winners in the left profile (assuming Anonymity and Neutrality)
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

Multiple-Districts Paradox



2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c





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So, any Condorcet consistent voting method violates the Multiple-Districts Property.

Referendum Paradox

D_1	D_2	D_3	D_4	D_5
Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

H. Nurmi (1998). *Voting paradoxes and referenda*. Social Choice and Welfare, Vol. 15, No. 3, pp. 333-350.

H. Dindar, G. Laffond and J. Laine (2017). *The strong referendum paradox*. Quality & Quantity: International Journal of Methodology, 51, pp. 1707 - 1731.

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- No is the majority outcome overall.

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Referendum Paradox

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Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

- No is the majority outcome overall.
- Yes wins a majority of the districts: The majority outcome in D_1, D_2, and D_3 is Yes and the majority outcome in D_4 and D_5 is No.

H. Nurmi (1998). *Voting paradoxes and referenda*. Social Choice and Welfare, Vol. 15, No. 3, pp. 333-350.

H. Dindar, G. Laffond and J. Laine (2017). *The strong referendum paradox*. Quality & Quantity: International Journal of Methodology, 51, pp. 1707 - 1731.

Electoral College

D. DeWitt and T. Schwartz (2016). *A Calamitous Compact*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 791 - 796.

J. R. Koza (2016). *A Not-So-Calamitous Compact: A Response to DeWitt and Schwartz*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 797 - 804.

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Condorcet Winner	–	–	–	–	✓	✓	✓
Condorcet Loser	–	✓	✓	✓	✓	–	✓
Monotonicity	✓	✓	–	–	✓	✓	✓
Positive Involvement	✓	✓	✓	–	–	✓	✓
Multiple Districts	✓	✓	–	–	–	–	–

Problem: There is no voting method that satisfies *all* of the principles of group decision making. So, how should you choose which voting method to use?

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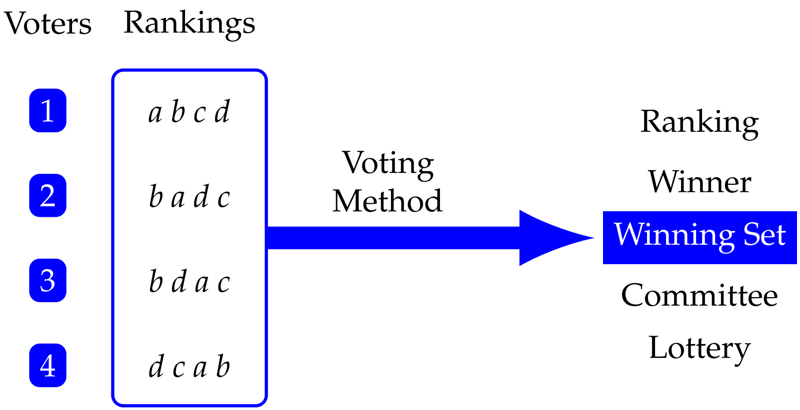
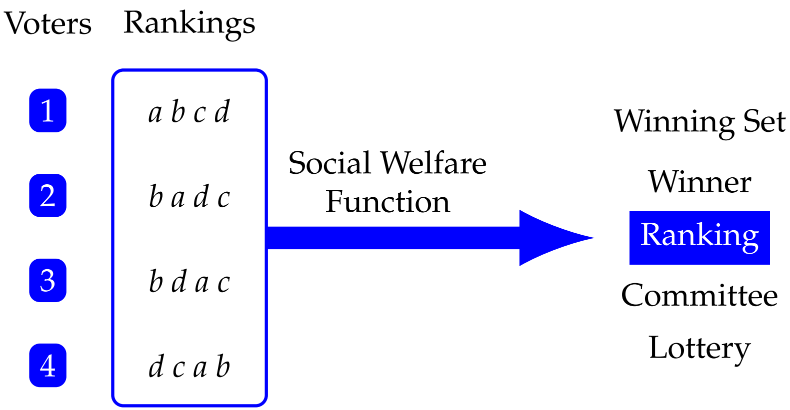
A fundamental result in social choice theory suggests that this situation is to be expected...


Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Voting Method
Ranking
Winner
Winning Set
Committee
Lottery


Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Social Welfare Function
Winning Set
Winner
Ranking
Committee
Lottery

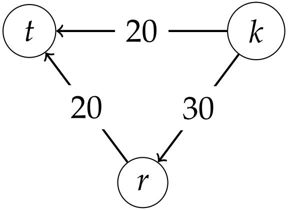
Social Welfare Functions

A **Social Welfare Function** f maps an election from a set $\mathcal{D}$ of possible elections to an ordering on the set of candidates.

Comments

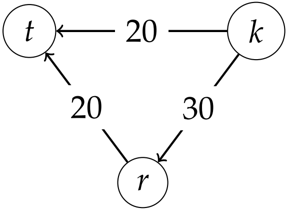
- $\mathcal{D}$ is called *domain* of the function f.
- Social Welfare Functions are *decisive*: every profile $\mathbf{P}$ in the domain is associated with exactly one ordering over the candidates
- For each profile $\mathbf{P}$, the ordering $f(\mathbf{P})$ is called the **social ordering** of $\mathbf{P}$ according to f.

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking
$k \, f(\mathbf{P}) \, r \, f(\mathbf{P}) \, t$

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k \ r \ t$

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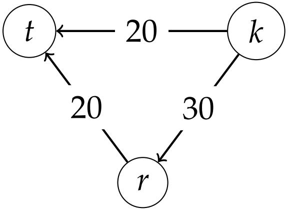

Social Ranking
$k \; r \; t$

Majority Ordering, Copeland, Borda

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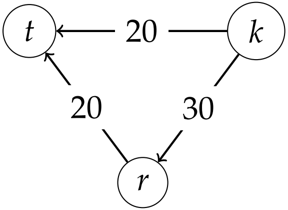

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Majority Ordering, Copeland, Borda

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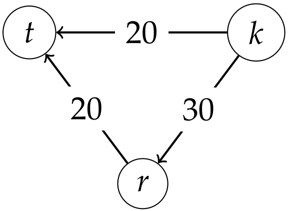
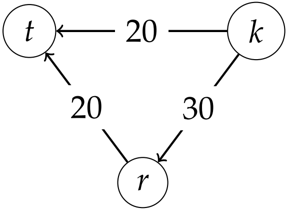


Social Ranking

$k \; r \; t$	Majority Ordering, Copeland, Borda
$k \; t \; r$	Minimize the maximum loss

40	35	25
t	r	k
k	k	r
r	t	t



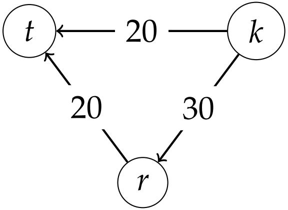
Social Ranking

$k \; r \; t$ Majority Ordering, Copeland, Borda

$k \; t \; r$ Minimize the maximum loss

$r \; t \; k$

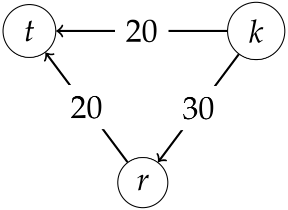
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k\ r\ t$	Majority Ordering, Copeland, Borda
$k\ t\ r$	Minimize the maximum loss
$r\ t\ k$	Instant Runoff

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

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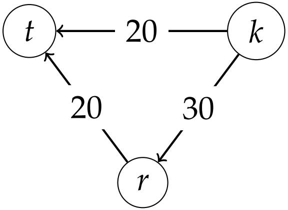
$r\ t\ k$	Instant Runoff
$t\ r\ k$	

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k \; r \; t$	Majority Ordering, Copeland, Borda
$k \; t \; r$	Minimize the maximum loss
$r \; t \; k$	Instant Runoff
$t \; r \; k$	Plurality scores

Examples

Borda Ordering: $\mathit{Borda}(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that the Borda score of a is greater than or equal to the Borda score for b in the profile $\mathbf{P}$.

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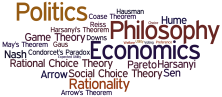
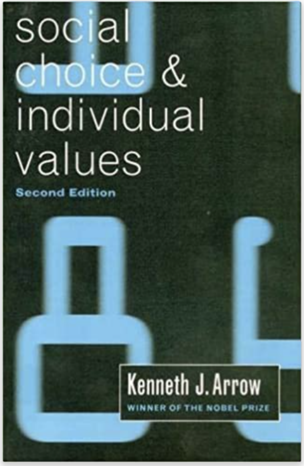
Plurality Ordering: $\mathit{Plurality}(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that the Plurality score of a is greater than or equal to the Plurality score for b.

Majority Ordering: $\mathit{Maj}(\mathbf{P})$ is the ordering where a is ranked above or tied with b provided that $\mathit{Margin}_{\mathbf{P}}(a, b) \geq 0$

Arrow's Impossibility Theorem

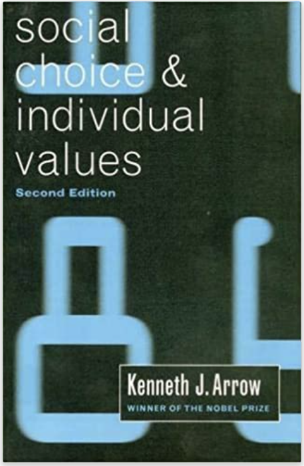




"For an area of study to become a recognized field, or even a recognized subfield, two things are required: It must be seen to have coherence, and it must be seen to have depth. The former often comes gradually, but the latter can arise in a single flash of brilliance....With social choice theory, there is little doubt as to the seminal result that made it a recognized field of study: Arrow's impossibility theorem."

A. Taylor, Social Choice and the Mathematics of Manipulation

Arrow's Impossibility Theorem



E. Maskin and A. Sen, editors (2014). *The Arrow Impossibility Theorem*. Columbia University Press.

M. Morreau (2019). *Arrow Impossibility Theorem*. Stanford Encyclopedia of Philosophy.

P. Suppes (2015). *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare 25(2), pp. 319-326.

Arrow's Axioms

Universal Domain

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"If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings."

(Arrow, p. 24)

Rationality

The social ranking is a **rational preference** on the set of candidates.

For all profile $\mathbf{P}$ in the domain of f, the ordering $f(\mathbf{P})$ is a complete and transitive ordering over the set of candidates.

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Example: Plurality and Borda always produces a complete and transitive ranking of the candidates, but the Majority ordering may output rankings that are not transitive.

Pareto/Unanimity

If each voter ranks *a* strictly above *b*, then so does the social ranking.

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For example, Plurality violates Pareto, but Borda and the Majority Ordering both satisfy Pareto.