# PHPE 400
# Individual and Group Decision Making

Eric Pacuit
University of Maryland
pacuit.org

| | Plurality | Borda | Instant Runoff | Coombs | Copeland | Minimax | MWSL |
|---|---|---|---|---|---|---|---|
| Anonymity | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Neutrality | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Pareto | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Condorcet Winner | — | — | — | — | ✓ | ✓ | ✓ |
| Condorcet Loser | — | ✓ | ✓ | ✓ | ✓ | — | ✓ |
| Monotonicity | ✓ | ✓ | — | — | ✓ | ✓ | ✓ |
| Immunity to Spoilers | — | — | — | — | — | ✓ | ✓ |
| Multiple Districts | ✓ | ✓ | — | — | — | — | — |

**Problem**: There is no voting method that satisfies *all* of the principles of group decision making. So, how should you choose which voting method to use?

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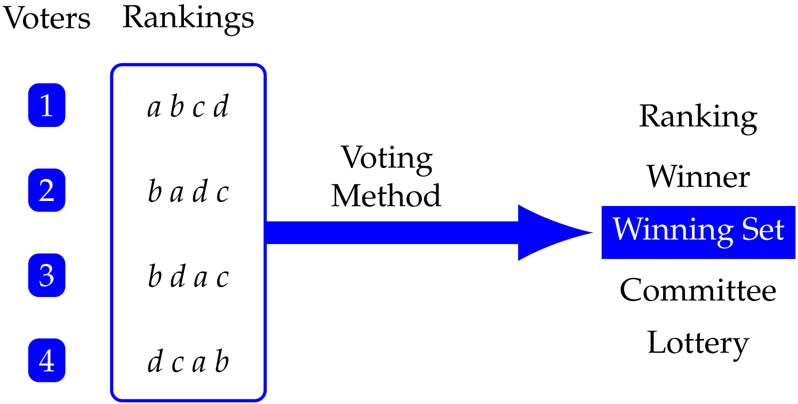
A fundamental result in social choice theory suggests that this situation is to be expected...


Voters
Rankings
1
2
3
4
a b c d
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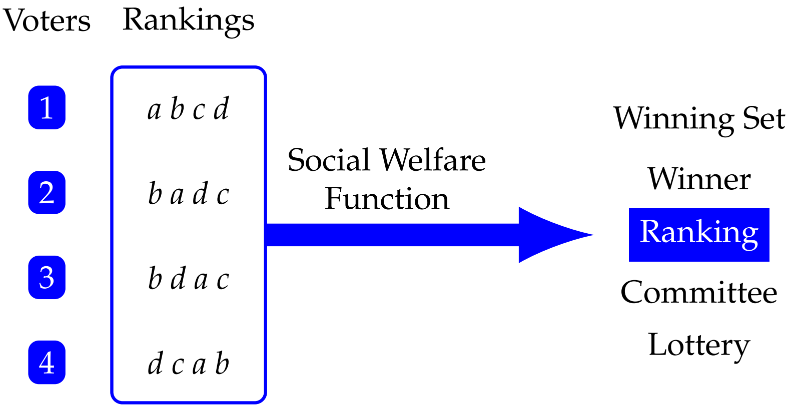
b d a c
d c a b
Voting
Method
Ranking
Winner
Winning Set
Committee
Lottery


Voters
Rankings
1
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a b c d
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b d a c
d c a b
Social Welfare
Function
Winning Set
Winner
Ranking
Committee
Lottery

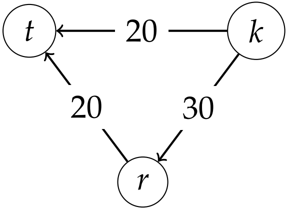
# Social Welfare Functions


A **Social Welfare Function** $f$ maps an election from a set $\mathcal{D}$ of possible elections to an ordering on the set of candidates.

**Comments**

- $\mathcal{D}$ is called *domain* of the function $f$.
- Social Welfare Functions are *decisive*: every profile $\mathbf{P}$ in the domain is associated with exactly one ordering over the candidates
- For each profile $\mathbf{P}$, the ordering $f(\mathbf{P})$ is called the **social ordering** of $\mathbf{P}$ according to $f$.

| 40 | 35 | 25 |
|---|---|---|
| $t$ | $r$ | $k$ |
| $k$ | $k$ | $r$ |
| $r$ | $t$ | $t$ |




Social Ranking

$k\, f(\mathbf{P})\, r\, f(\mathbf{P})\, t$

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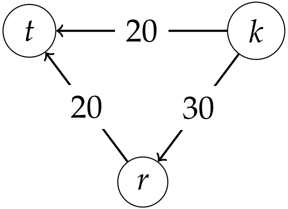


Social Ranking

$k \ r \ t$

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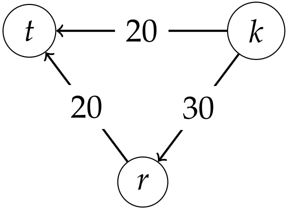


Social Ranking
$k\ \ r\ \ t$

Majority Ordering, Copeland, Borda

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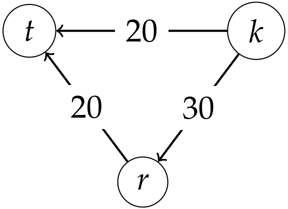


Social Ranking

$k \ r \ t$

$k \ t \ r$

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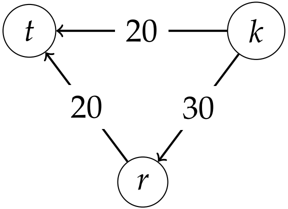


Social Ranking

$k \; r \; t$ Majority Ordering, Copeland, Borda

$k \; t \; r$ Minimize the maximum loss

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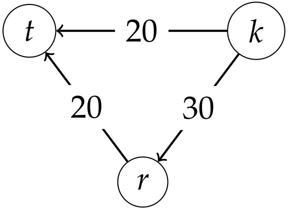


Social Ranking

$k\ r\ t$ — Majority Ordering, Copeland, Borda

$k\ t\ r$ — Minimize the maximum loss

$r\ t\ k$

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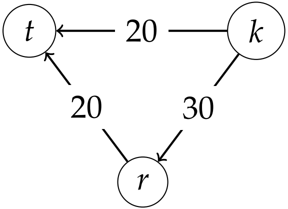


Social Ranking

| | |
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| $k\ r\ t$ | Majority Ordering, Copeland, Borda |
| $k\ t\ r$ | Minimize the maximum loss |
| $r\ t\ k$ | Instant Runoff Removal Order |

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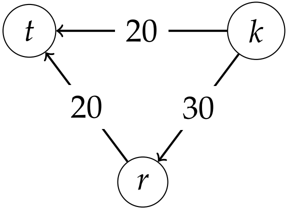


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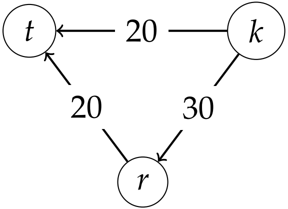


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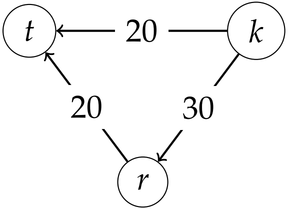


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| $t\ r\ k$ | |

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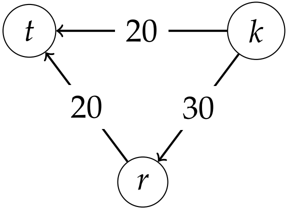


Social Ranking

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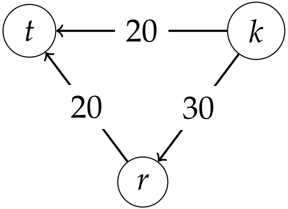


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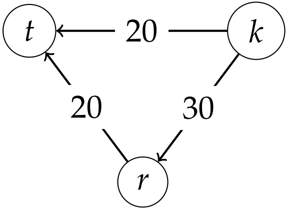


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| $t$ $r$ $k$ | Plurality scores |
| $t$ $k$ $r$ | Iterative Plurality |

# Examples

**Borda Ordering**: *Borda*(**P**) is the ordering where $a$ is ranked above or tied with $b$ provided that the Borda score of $a$ is greater than or equal to the Borda score for $b$ in the profile **P**.

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**Plurality Ordering**: *Plurality*($\mathbf{P}$) is the ordering where $a$ is ranked above or tied with $b$ provided that the Plurality score of $a$ is greater than or equal to the Plurality score for $b$.

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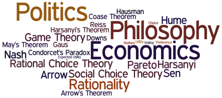
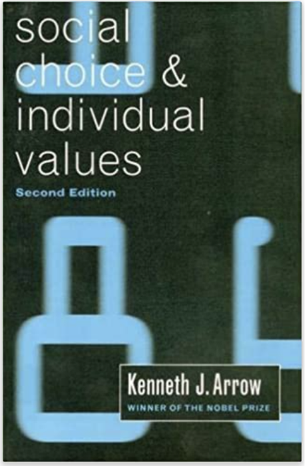
**Plurality Ordering**: $Plurality(\mathbf{P})$ is the ordering where $a$ is ranked above or tied with $b$ provided that the Plurality score of $a$ is greater than or equal to the Plurality score for $b$.

**Majority Ordering**: $Maj(\mathbf{P})$ is the ordering where $a$ is ranked above or tied with $b$ provided that $Margin_{\mathbf{P}}(a, b) \geq 0$

# Arrow’s Impossibility Theorem

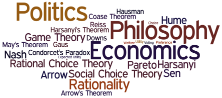


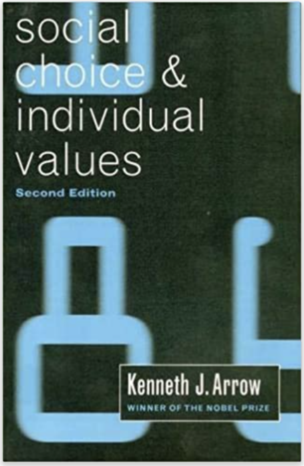




“For an area of study to become a recognized field, or even a recognized subfield, two things are required: It must be seen to have coherence, and it must be seen to have depth. The former often comes gradually, but the latter can arise in a single flash of brilliance....With social choice theory, there is little doubt as to the seminal result that made it a recognized field of study: **Arrow’s impossibility theorem**.”

A. Taylor, Social Choice and the Mathematics of Manipulation

# Arrow's Impossibility Theorem





E. Maskin and A. Sen, editors (2014). *The Arrow Impossibility Theorem*. Columbia University Press.

M. Morreau (2019). *Arrow Impossibility Theorem*. Stanford Encyclopedia of Philosophy.

P. Suppes (2015). *The pre-history of Kenneth Arrow's social choice and individual values*. Social Choice and Welfare 25(2), pp. 319-326.

# Arrow’s Axioms

# Universal Domain

Voter’s are free to choose any ranking, and the voters’ choices are independent.

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"If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings."

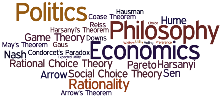
(Arrow, p. 24)

# Rationality

The social ranking is a **rational preference** on the set of candidates.

For all profile $\mathbf{P}$ in the domain of $f$, the ordering $f(\mathbf{P})$ is a complete and transitive ordering over the set of candidates.

# Rationality

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Example: Plurality and Borda always produces a complete and transitive ranking of the candidates, but **the Majority ordering may output rankings that are not transitive**.

# Pareto/Unanimity

If each voter ranks *a* strictly above *b*, then so does the social ranking.

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For example, Plurality violates Pareto, but Borda and the Majority Ordering both satisfy Pareto.

# Voting Splitting

| 40 | 35 | 25 |
|---|---|---|
| $t$ | $r$ | $k$ |
| $k$ | $k$ | $t$ |
| $r$ | $t$ | $r$ |

According to Plurality, $t$ wins and $k$ loses...
even though a majority of voters prefer $k$ to $t$.

$r$ **splits the vote** of all voters rankings $k$ above $t$.

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Independence of Irrelevant Alternatives: If $k$ wins and $t$ loses in the profile on the right, then the same should happen in the profile on the left

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The social ranking (higher, lower, or indifferent) of two alternatives $a$ and $b$ depends only the relative rankings of $a$ and $b$ for each voter.

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For all profiles $\mathbf{P}$ and $\mathbf{P}'$:

$$\text{If } \mathbf{P}_{i\{a,b\}} = \mathbf{P}'_{i\{a,b\}} \text{ for all } i \in V, \text{ then } f(\mathbf{P})_{\{a,b\}} = f(\mathbf{P}')_{\{a,b\}}.$$

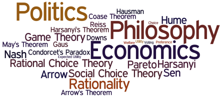
where $P_{\{x,y\}}$ is the ranking on $x$ and $y$ defined as follows:

$$P_{\{x,y\}} = P \cap \{x,y\} \times \{x,y\}$$

# Independence of Irrelevant Alternatives

(IIA): For all profiles $\mathbf{P}, \mathbf{P}'$ and $x, y \in X$,
if $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then $f(\mathbf{P})_{\{x,y\}} = f(\mathbf{P}')_{\{x,y\}}$.

# Independence of Irrelevant Alternatives




(IIA): For all profiles $\mathbf{P}, \mathbf{P}'$ and $x, y \in X$,
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(IIA): For all profiles $\mathbf{P}$ and all $x, y \in X$,
if $\mathbf{P}'$ is a profile in the domain of $f$ such that $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then

- If $x$ defeats $y$ according to $f$ in $\mathbf{P}$, then $x$ defeats $y$ according to $f$ in $\mathbf{P}'$
- If $x$ does not defeat $y$ according to $f$ in $\mathbf{P}$, then $x$ does not defeat $y$ according to $f$ in $\mathbf{P}'$

# Borda violates IIA, Example 1

**P**:

| 45 | 55 | $f_{borda}(\mathbf{P})$ |
|---|---|---|
| $a$ | $b$ | $a$ |
| $c$ | $a$ | $b$ |
| $b$ | $c$ | $c$ |

**P′**:

| 45 | 55 | $f_{borda}(\mathbf{P'})$ |
|---|---|---|
| $a$ | $b$ | $b$ |
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$\mathbf{P}$:

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$\mathbf{P}'$:

| 45 | 55 | $f_{borda}(\mathbf{P}')$ |
|---|---|---|
| $a$ | $b$ | $b$ |
| $b$ | $a$ | $a$ |
| $c$ | $c$ | $c$ |

$\mathbf{P}_{|\{a,b\}} = \mathbf{P}'_{|\{a,b\}}$, but
$a$ beats $b$ in $\mathbf{P}$ according to Borda, and $b$ beats $a$ in $\mathbf{P}'$ according to Borda.

# Borda violates IIA, Example 2

**P**:

| 1 | 1 | $f_{borda}(\mathbf{P})$ |
|---|---|---|
| $a$ | $c$ | $a\ b\ c$ |
| $b$ | $b$ | $d$ |
| $c$ | $a$ | |
| $d$ | $d$ | |

**P′**:

| 1 | 1 | $f_{borda}(\mathbf{P}')$ |
|---|---|---|
| $a$ | $c$ | $a\ b$ |
| $b$ | $b$ | $c$ |
| $d$ | $a$ | $d$ |
| $c$ | $d$ | |

# Borda violates IIA, Example 2

**P**:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $b$ |
| $c$ | $a$ |
| $d$ | $d$ |

| $f_{borda}(\mathbf{P})$ |
|---|
| $a$ $b$ $c$ |
| $d$ |

**P′**:

| 1 | 1 |
|---|---|
| $a$ | $c$ |
| $b$ | $b$ |
| $d$ | $a$ |
| $c$ | $d$ |

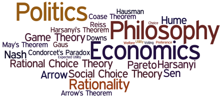
| $f_{borda}(\mathbf{P}')$ |
|---|
| $a$ $b$ |
| $c$ |
| $d$ |

$\mathbf{P}_{|\{b,c\}} = \mathbf{P}'_{|\{b,c\}}$, but

$b$ and $c$ are tied in **P** according to Borda,

and $b$ is ranked above $c$ in $\mathbf{P}'$ accodring to Borda.

# Dictatorship




A voter $d \in V$ is a **dictator** for $f$ if society strictly prefers $a$ over $b$ according to $f$ *whenever* $d$ strictly prefers $a$ over $b$.

# Dictatorship

A voter $d \in V$ is a **dictator** for $f$ if society strictly prefers $a$ over $b$ according to $f$ *whenever* $d$ strictly prefers $a$ over $b$.

There is a $d \in V$ such that for each profile $\mathbf{P}$, if $a\ \mathbf{P}_d\ b$ then
$a$ is strictly preferred to $b$ according to $f(\mathbf{P})$

Non-Dictatorship: There is no voter that is a dictator for $f$.

# Summary

- Every social welfare functions that we have discussed satisfies universal domain and non-dictatorship.
- Most social welfare functions satisfies Pareto (except ranking by Plurality scores).
- Some social welfare functions satisfy Rationality: e.g., ranking by Plurality scores and ranking by Borda scores
- Some social welfare functions satisfy IIA: e.g., Majority ordering,

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Are there any social welfare functions that satisfy *all* of Arrow's axioms?

# Arrow's Theorem

**Theorem** (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies Universal Domain, Rationality, Pareto, Independence of Irrelevant Alternatives (IIA) is a Dictatorship.

- Alternative statement of the theorem: Suppose that there are at least three candidates and finitely many voters. There is no social welfare function that satisfies Universal Domain, Rationality, Pareto, Independence of Irrelevant Alternatives (IIA), and Non-Dictatorship.