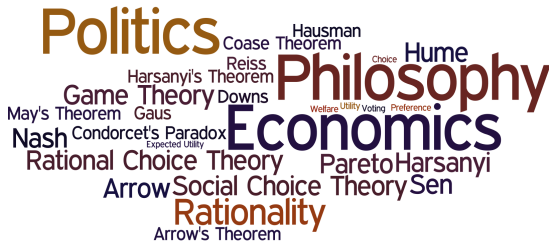


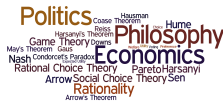
PHPE 400

Individual and Group Decision Making

Eric Pacuit
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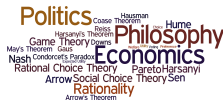


Anonymity and Neutrality



- ▶ F satisfies **Anonymity**: permuting the voters does not change the set of winners.
- ▶ F satisfies **Neutrality**: permuting the candidates results in a winning set that is permuted in the same way.

Anonymity and Neutrality

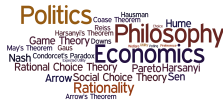


- ▶ F satisfies **Anonymity**: permuting the voters does not change the set of winners.
- ▶ F satisfies **Neutrality**: permuting the candidates results in a winning set that is permuted in the same way.

$\implies$ in 2-candidate profiles, if the same number of voters rank a above b as b above a , then $a \in F(\mathbf{P})$ if, and only if, $b \in F(\mathbf{P})$

(a wins according to F if and only if b wins according to F).

Weak Positive Responsiveness



- F satisfies **Weak Positive Responsiveness** if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, if
1. $a \in F(\mathbf{P})$ (a is a winner in $\mathbf{P}$ according to F) and
 2. $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter who ranked a uniquely last in $\mathbf{P}$ switching to ranking a uniquely first in $\mathbf{P}'$ (while all other voters' rankings are unchanged),
- then $F(\mathbf{P}') = \{a\}$ (a is the **unique** winner in $\mathbf{P}'$ according to F).

Below is all possible profiles for 3 voters and two candidates a and b , and the outcomes of different voting methods.

Profile	Voter 1	Always a	Minority	Unanimity	Majority
$(a \succ b, a \succ b, a \succ b)$	a	a	b	a	a
$(a \succ b, a \succ b, b \succ a)$	a	a	b	a, b	a
$(a \succ b, b \succ a, a \succ b)$	a	a	b	a, b	a
$(a \succ b, b \succ a, b \succ a)$	a	a	a	a, b	b
$(b \succ a, a \succ b, a \succ b)$	b	a	b	a, b	a
$(b \succ a, a \succ b, b \succ a)$	b	a	a	a, b	b
$(b \succ a, b \succ a, a \succ b)$	b	a	a	a, b	b
$(b \succ a, b \succ a, b \succ a)$	b	a	a	b	b

Profile	Voter 1	Always a	Minority	Unanimity	Majority
$(a P b, a P b, a P b)$	a	a	b	a	a
$(a P b, a P b, b P a)$	a	a	b	a, b	a
$(a P b, b P a, a P b)$	a	a	b	a, b	a
$(a P b, b P a, b P a)$	a	a	a	a, b	b
$(b P a, a P b, a P b)$	b	a	b	a, b	a
$(b P a, a P b, b P a)$	b	a	a	a, b	b
$(b P a, b P a, a P b)$	b	a	a	a, b	b
$(b P a, b P a, b P a)$	b	a	a	b	b

Voter 1 violates Anonymity: The method **Voter 1** assigns different winners to the profiles $(a P b, a P b, b P a)$ and $(b P a, a P b, a P b)$.

Profile	Voter 1	Always a	Minority	Unanimity	Majority
$(a P b, a P b, a P b)$	a	a	b	a	a
$(a P b, a P b, b P a)$	a	a	b	a, b	a
$(a P b, b P a, a P b)$	a	a	b	a, b	a
$(a P b, b P a, b P a)$	a	a	a	a, b	b
$(b P a, a P b, a P b)$	b	a	b	a, b	a
$(b P a, a P b, b P a)$	b	a	a	a, b	b
$(b P a, b P a, a P b)$	b	a	a	a, b	b
$(b P a, b P a, b P a)$	b	a	a	b	b

Always a violates Neutrality: The method **Always a** assigns a as a winner to $(a P b, a P b, b P a)$, and assigns a to $(b P a, b P a, a P b)$ (but it should assign b according to Neutrality).

Profile	Voter 1	Always a	Minority	Unanimity	Majority
$(a P b, a P b, a P b)$	a	a	b	a	a
$(a P b, a P b, b P a)$	a	a	b	a, b	a
$(a P b, b P a, a P b)$	a	a	b	a, b	a
$(a P b, b P a, b P a)$	a	a	a	a, b	b
$(b P a, a P b, a P b)$	b	a	b	a, b	a
$(b P a, a P b, b P a)$	b	a	a	a, b	b
$(b P a, b P a, a P b)$	b	a	a	a, b	b
$(b P a, b P a, b P a)$	b	a	a	b	b

Minority violates Weak Positive Responsiveness: b is a winner in $(b P a, a P b, a P b)$, but $(b P a, b P a, a P b)$ is a profile in which one voter (voter 2) moves b from the bottom to the top of their ranking yet b does not win in this profile.

Profile	Voter 1	Always a	Minority	Unanimity	Majority
$(a P b, a P b, a P b)$	a	a	b	a	a
$(a P b, a P b, b P a)$	a	a	b	a, b	a
$(a P b, b P a, a P b)$	a	a	b	a, b	a
$(a P b, b P a, b P a)$	a	a	a	a, b	b
$(b P a, a P b, a P b)$	b	a	b	a, b	a
$(b P a, a P b, b P a)$	b	a	a	a, b	b
$(b P a, b P a, a P b)$	b	a	a	a, b	b
$(b P a, b P a, b P a)$	b	a	a	b	b

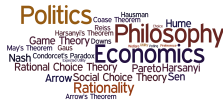
Unanimity violates Weak Positive Responsiveness: b is a winner in $(b P a, a P b, a P b)$, but $(b P a, b P a, a P b)$ is a profile in which one voter (voter 2) moves b from the bottom to the top of their ranking yet b is **not the unique winner** in this profile.

	Anonymity	Neutrality	Weak Positive Responsiveness
Voter 1	✗	✓	✗
Always a	✓	✗	✓
Minority Rule	✓	✓	✗
Unanimity Rule	✓	✓	✗
Majority Rule	✓	✓	✓

Let F be a voting method on the domain of two-alternative profiles. Then the following are equivalent:

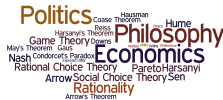
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Proof Sketch



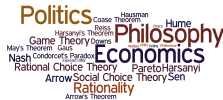
Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.
Can we have $F(a P b, a P b, b P a) = \{b\}$?

Proof Sketch



Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.
Can we have $F(a P b, a P b, b P a) = \{b\}$? No!

Proof Sketch



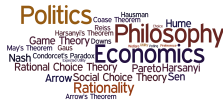
Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.

Can we have $F(a P b, a P b, b P a) = \{b\}$? No!

Suppose that $F(a P b, a P b, b P a) = \{b\}$

By Neutrality, $F(b P a, b P a, a P b) = \{a\}$

Proof Sketch



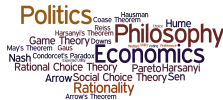
Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.
Can we have $F(a P b, a P b, b P a) = \{b\}$? No!

Suppose that $F(a P b, a P b, b P a) = \{b\}$

By Neutrality, $F(b P a, b P a, a P b) = \{a\}$

By Anonymity, $F(a P b, b P a, b P a) = \{a\}$

Proof Sketch



Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.
Can we have $F(a P b, a P b, b P a) = \{b\}$? No!

Suppose that $F(a P b, a P b, b P a) = \{b\}$

By Neutrality, $F(b P a, b P a, a P b) = \{a\}$

By Anonymity, $F(a P b, b P a, b P a) = \{a\}$

By Weak Positive Responsiveness, $F(a P b, a P b, b P a) = \{a\}$

Suppose that $F(a \mathrel{P} b, a \mathrel{P} b, b \mathrel{P} a) = \{b\}$

By Neutrality, $F(b P a, b P a, a P b) = \{a\}$

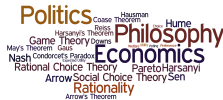
By Anonymity, $F(a P b, b P a, b P a) = \{a\}$

By Weak Positive Responsiveness, $F(a P b, a P b, b P a) = \{a\}$

Contradiction: Since F is a function, we can't have:

$$F(a P b, a P b, b P a) = \{a, b\} \text{ and } F(a P b, a P b, b P a) = \{a\}$$

Proof Sketch



Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness.
Can we have $F(a P b, a P b, b P a) = \{a, b\}$? No!

Suppose that $F(a P b, a P b, b P a) = \{a, b\}$

By Neutrality, $F(b P a, b P a, a P b) = \{a, b\}$

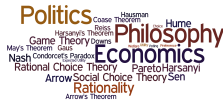
By Anonymity, $F(a P b, b P a, b P a) = \{a, b\}$

By Weak Positive Responsiveness, $F(a P b, a P b, b P a) = \{a\}$

Contradiction: Since F is a function, we can't have:

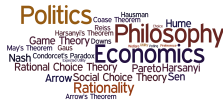
$$F(a P b, a P b, b P a) = \{a, b\} \text{ and } F(a P b, a P b, b P a) = \{a\}$$

Justifying Majority Rule



May's Theorem is a **proceduralist** justification of majority rule showing that Majority Rule is the unique group decision method satisfying two basic principles of fairness (Anonymity and Neutrality) and a basic principle ensuring that the outcome responds appropriately to the voters' opinions (Weak Positive Responsiveness).

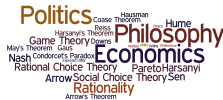
Justifying Majority Rule



May's Theorem is a **proceduralist** justification of majority rule showing that Majority Rule is the unique group decision method satisfying two basic principles of fairness (Anonymity and Neutrality) and a basic principle ensuring that the outcome responds appropriately to the voters' opinions (Weak Positive Responsiveness).

We can also give an **epistemic justification** of majority rule showing that has a high probability of identifying the **correct answer to a question**.

Justifying Majority Rule II



In many group decision making problems, one of the alternatives is the *correct* one.

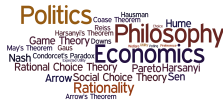
Which group decision making method is best for finding the “correct” alternative?

The Condorcet Jury Theorem



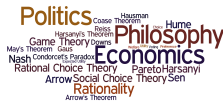
<https://cjt-tutorial.streamlit.app/>

Condorcet Jury Theorem



- ▶ $V = \{1, 2, \dots, n\}$ is the set of experts.
- ▶ $\{0, 1\}$ is the set of outcomes.
- ▶ $\mathbf{x}$ be a random variable (called the **state**) whose values range over the two outcomes. We write $\mathbf{x} = 1$ when the outcome is 1 and $\mathbf{x} = 0$ when the outcome is 0.
- ▶ $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are random variables representing the votes for experts $1, 2, \dots, n$. For each $i = 1, \dots, n$, we write $\mathbf{v}_i = 1$ when expert i 's vote is 1 and $\mathbf{v}_i = 0$ when expert i 's vote is 0.
- ▶ R_i is the event that expert i votes correctly: it is the event that $\mathbf{v}_i$ coincides with $\mathbf{x}$ (i.e., $\mathbf{v}_i = 1$ and $\mathbf{x} = 1$ or $\mathbf{v}_i = 0$ and $\mathbf{x} = 0$).

Condorcet Jury Theorem

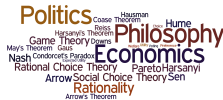


Independence: The correctness events $R_1, R_2, \dots, R_n$ are independent.

Competence: The experts' competences $Pr(R_i)$ (i) exceeds $\frac{1}{2}$ and (ii) is the same for each voter i .

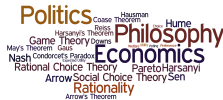
Condorcet Jury Theorem: Assume Independence and Competence. Then, as the group size increases, the probability of that the majority is correct (i) increases (growing reliability), and (ii) tends to one (infallibility).

Justifying Majority Rule



May's Theorem is a **proceduralist** justification of majority rule showing that Majority Rule is the unique group decision method satisfying two basic principles of fairness (Anonymity and Neutrality) and a basic principle ensuring that the outcome responds appropriately to the voters' opinions (Weak Positive Responsiveness).

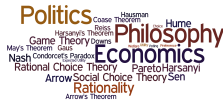
Justifying Majority Rule



May's Theorem is a **proceduralist** justification of majority rule showing that Majority Rule is the unique group decision method satisfying two basic principles of fairness (Anonymity and Neutrality) and a basic principle ensuring that the outcome responds appropriately to the voters' opinions (Weak Positive Responsiveness).

The Condorcet Jury Theorem is an **epistemic** justification of majority rule showing that under the assumption that the voters are *competent* in the sense that each voter has a greater than 50% chance of voting correctly and that the events that the voters are correct are independent, then the probability that the majority is correct increases to 1 as the size of the group increases.

Beyond Two Candidates

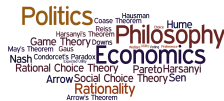


With 2 candidates, Majority Rule is uniquely justified (May's Theorem, Condorcet Jury Theorem)

With more than 2 candidates, there are two problems:

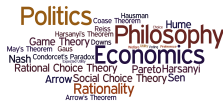
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Multiple Elections Paradox



S. Brams, D. M. Kilgour, and W. Zwicker (1998). *The paradox of multiple elections*. Social Choice and Welfare, 15(2), pp. 211 - 236.

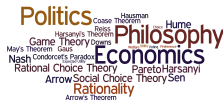
Multiple Elections Paradox



Voters are asked to give their opinion on three yes/no issues:

YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
1	1	1	3	1	3	3	0

Multiple Elections Paradox



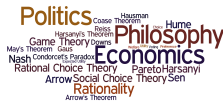
Voters are asked to give their opinion on three yes/no issues:

YYY	YYN	YNY	YNN	YYY	YYN	YNY	YNN
1	1	1	3	1	3	3	0

Outcome by majority vote

Proposition 1: *N* (7 - 6)

Multiple Elections Paradox



Voters are asked to give their opinion on three yes/no issues:

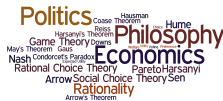
Y ₁ Y ₂ Y ₃	Y ₁ Y ₂ N ₃	Y ₁ N ₂ Y ₃	Y ₁ N ₂ N ₃	N ₁ Y ₂ Y ₃	N ₁ Y ₂ N ₃	N ₁ N ₂ Y ₃	N ₁ N ₂ N ₃
1	1	1	3	1	3	3	0

Outcome by majority vote

Proposition 1: N (7 - 6)

Proposition 2: N (7 - 6)

Multiple Elections Paradox



Voters are asked to give their opinion on three yes/no issues:

YY ^Y	YY ^N	YN ^Y	YN ^N	NY ^Y	NY ^N	NN ^Y	NN ^N
1	1	1	3	1	3	3	0

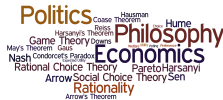
Outcome by majority vote

Proposition 1: N (7 - 6)

Proposition 2: N (7 - 6)

Proposition 3: N (7 - 6)

Multiple Elections Paradox



Voters are asked to give their opinion on three yes/no issues:

YYY	YYN	YNY	YNN	NYY	NYN	NNY	NNN
1	1	1	3	1	3	3	0

Outcome by majority vote

Proposition 1: N (7 - 6)

Proposition 2: N (7 - 6)

Proposition 3: N (7 - 6)

But there is no support for NNN!

S. Brams, M. Kilgour and W. Zwicker (1997). *Voting on referenda: the separability problem and possible solutions*. Electoral Studies, 16(3), pp. 359 - 377.

D. Lacy and E. Niou (2000). *A problem with referenda*. Journal of Theoretical Politics 12(1), pp. 5 - 31.

J. Lang and L. Xia (2009). *Sequential composition of voting rules in multi-issue domains*. Mathematical Social Sciences 57(3), pp. 304 - 324.

L. Xia, V. Conitzer and J. Lang (2010). *Strategic Sequential Voting in Multi-Issue Domains and Multiple-Election Paradoxes*. In Proceedings of the Twelfth ACM Conference on Electronic Commerce (EC-11), pp. 179-188.

“Is a conflict between the proposition and combination winners necessarily bad?

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“Is a conflict between the proposition and combination winners necessarily bad? ... The paradox does not just highlight problems of aggregation and packaging, however, but strikes at the core of social choice—both what it means and how to uncover it. In our view, the paradox shows there may be a clash between two different meanings of social choice, leaving unsettled the best way to uncover what this elusive quantity is.” (pg. 234).

S. Brams, D. M. Kilgour, and W. Zwicker (1998). *The paradox of multiple elections*. Social Choice and Welfare, 15(2), pp. 211 - 236.

Judgement Aggregation Paradoxes

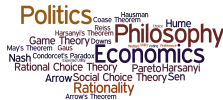


Kornhauser and Sager. *Unpacking the court*. Yale Law Journal, 1986.

P. Mongin. *The doctrinal paradox, the discursive dilemma, and logical aggregation theory*. Theory and Decision, 73(3), pp 315 - 355, 2012.

C. List and P. Pettit. *Aggregating sets of judgments: An impossibility result*. Economics and Philosophy 18, pp. 89 - 110, 2002.

Judgement Aggregation Paradox



Should we hire the candidate?

- ▶ Is the candidate good at research (r)?
- ▶ Is the candidate good at teaching (t)?
- ▶ We should hire the candidate if and only if the candidate is good at research and teaching. ($r \wedge t$)

Judgement Aggregation Paradox



Is the candidate good at research (r)? Is the candidate good at teaching (t)?
Should we hire the candidate (h)?

	r	t		h
Voter 1				
Voter 2				
Voter 3				
Group				

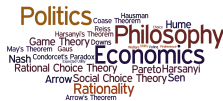
Should we hire the candidate (h)?

	r	t		
Voter 1	Yes	Yes		
Voter 2	Yes	No		
Voter 3	No	Yes		
Group	Yes	Yes		

Should we hire the candidate (h)?

	r	t	$(r \wedge t) \leftrightarrow h$	h
Voter 1	Yes	Yes		
Voter 2	Yes	No		
Voter 3	No	Yes		
Group	Yes	Yes	Yes	Yes

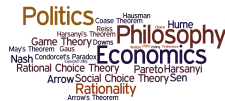
Judgement Aggregation Paradox



Is the candidate good at research (r)? Is the candidate good at teaching (t)?
Should we hire the candidate (h)?

	r	t	$(r \wedge t) \leftrightarrow h$	h
Voter 1	Yes	Yes	Yes	Yes
Voter 2	Yes	No	Yes	No
Voter 3	No	Yes	Yes	No
Group				No

Judgement Aggregation Paradox



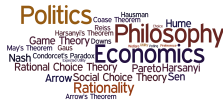
Is the candidate good at research (r)? Is the candidate good at teaching (t)?
Should we hire the candidate (h)?

	r	t	$(r \wedge t) \leftrightarrow h$	h
Voter 1	Yes	Yes	Yes	Yes
Voter 2	Yes	No	Yes	No
Voter 3	No	Yes	Yes	No
Group	Yes	Yes	Yes	Y/N

With more than 2 candidates, there are two problems:

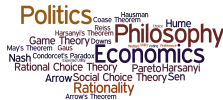
- ## What additional principles should we use to distinguish between the different voting methods?

Principles of group decision making



- ▶ **Anonymity:** If voters swap their ballots, then the outcome is unaffected.
- ▶ **Neutrality:** If candidates are exchanged in every ranking, then the outcome changes accordingly.
- ▶ **Resoluteness:** Always elect a single winner.

Condorcet Triples and Resoluteness



n	n	n
a	b	c
b	c	a
c	a	b

n	n	n
a	c	b
c	b	a
b	a	c

Fact. In both profiles, any voting method satisfying anonymity and neutrality must select all candidates as winners

1	1	1
a	b	c
b	c	a
c	a	b

Consider $\mathbf{P} = (a\ b\ c, b\ c\ a, c\ a\ b)$ and suppose that $F(a\ b\ c, b\ c\ a, c\ a\ b) = \{a\}$

Suppose that $F(\textcolor{blue}{a} \textcolor{red}{b} \textcolor{green}{c}, \textcolor{red}{b} \textcolor{green}{c} \textcolor{blue}{a}, \textcolor{green}{c} \textcolor{blue}{a} \textcolor{red}{b}) = \{\textcolor{blue}{a}\}$

Suppose that $F(\boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a}, \boxed{c} \boxed{a} \boxed{b}) = \{\boxed{a}\}$

1. Swap a and b in everyone's rankings in the given profile. Then, by Neutrality:

$$F(\boxed{b} \boxed{a} \boxed{c}, \boxed{a} \boxed{c} \boxed{b}, \boxed{c} \boxed{b} \boxed{a}) = \{\boxed{b}\}$$

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2. Swap b and c in everyone's rankings in the profile from step 1. Then, by Neutrality:

$$F(\boxed{c} \boxed{a} \boxed{b}, \boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a}) = \{\boxed{c}\}$$

Suppose that $F(\boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a}, \boxed{c} \boxed{a} \boxed{b}) = \{\boxed{a}\}$

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2. Swap b and c in everyone's rankings in the profile from step 1. Then, by Neutrality:

$$F(\boxed{c} \boxed{a} \boxed{b}, \boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a}) = \{\boxed{c}\}$$

3. By Anonymity, the original profile and the profile in step 3 must have the same winners:

$$F(\boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a}, \boxed{c} \boxed{a} \boxed{b}) = F(\boxed{c} \boxed{a} \boxed{b}, \boxed{a} \boxed{b} \boxed{c}, \boxed{b} \boxed{c} \boxed{a})$$

Suppose that $F(\text{a b c}, \text{b c a}, \text{c a b}) = \{\text{a}\}$

1. Swap a and b in everyone's rankings in the given profile. Then, by Neutrality:

$$F(\text{b a c}, \text{a c b}, \text{c b a}) = \{\text{b}\}$$

2. Swap b and c in everyone's rankings in the profile from step 1. Then, by Neutrality:

$$F(\text{c a b}, \text{a b c}, \text{b c a}) = \{\text{c}\}$$

3. By Anonymity, the original profile and the profile in step 3 must have the same winners:

$$F(\text{a b c}, \text{b c a}, \text{c a b}) = F(\text{c a b}, \text{a b c}, \text{b c a})$$

4. 1 and 2 contradict 3 since

$$F(\text{a b c}, \text{b c a}, \text{c a b}) = \{\text{a}\} \neq \{\text{c}\} = F(\text{c a b}, \text{a b c}, \text{b c a}).$$

So, tie-breaking cannot be built-in to a voting method: there is no voting method that satisfies Anonymity, Neutrality and always elects a single winner.