PHPE 400
Individual and Group Decision Making

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A **rational** preference over lotteries involves more than the assumption that the decision maker's preferences are transitive and complete:

1. Independence axiom
2. Compound lottery axiom
3. Continuity axiom

Ordinal Utility and Expected Utility

Suppose that $X = \{a, b, c\}$ and the decision maker has the strict preference

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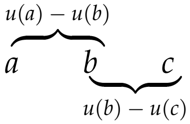

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	a	b	c		
u_1	4	1.5	1	$u_1(a) > u_1(b) > u_1(c)$	$EU(L_1, u_1) > EU(L_2, u_1)$
u_2	4	2.5	1	$u_2(a) > u_2(b) > u_2(c)$	$EU(L_1, u_2) = EU(L_2, u_2)$
u_3	4	3	1	$u_3(a) > u_3(b) > u_3(c)$	$EU(L_1, u_3) < EU(L_2, u_3)$

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Problem: u_1, u_2, and u_3 each represent the decision maker's preferences, but rank L_1 and L_2 differently according to the expected utility.

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> **Ratio scale**: Quantitative comparisons of objects, accurately reflects ratios between objects. E.g., 10lb (= 4.53592kg) is twice as much as 5lb (= 2.26796kg).

Measuring Utility

L. Narens and B. Skyrms (2020). *The Pursuit of Happiness: Philosophical and Psychological Foundations of Utility*. Oxford University Press.

I. Moscati (2018). *Measuring Utility From the Marginal Revolution to Behavioral Economics*. Oxford University Press.

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1. Prefer lotteries that are closer to 50-50:

$$U_1([a : r, b : (1 - r)]) = -|r - \frac{1}{2}|$$

2. Prefer lotteries with a higher chance of ending up with a:

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The second preference is *rational* while the first preference is irrational: Intuitively, preferences over lotteries should have something to do with preferences over consequences.

A function $U : \mathcal{L} \to \mathbb{R}$ is **linear** provided that for any $L = [x_1 : p_1, \ldots, x_n : p_n]$,

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U_2 is linear: For any lottery $[a : r, b : (1 - r)]$,

$$\begin{aligned} U_2([a : r, b : 1 - r]) &= \boxed{r} \\ rU_2([a : 1]) + (1 - r)U_2([b : 1]) &= r \times 1 + (1 - r) \times 0 \\ &= \boxed{r} \end{aligned}$$

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U_1 is **not** linear: Consider the lottery $[a : \frac{1}{4}, b : \frac{3}{4}]$

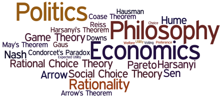
$$\begin{aligned} U_1([a : \tfrac{1}{4}, b : \tfrac{3}{4}]) &= -|\tfrac{1}{4} - \tfrac{1}{2}| \\ &= -\tfrac{1}{4} \\ \tfrac{1}{4}U_1([a : 1]) + \tfrac{3}{4}U_1([b : 1]) &= \tfrac{1}{4} \times -|1 - \tfrac{1}{2}| + \tfrac{3}{4} \times -|0 - \tfrac{1}{2}| \\ &= -\tfrac{1}{8} + -\tfrac{3}{8} \\ &= -\tfrac{1}{2} \end{aligned}$$

Given a rational preference (P, I) over the set of lotteries $\mathcal{L}$ we want to guarantee that the rational preference is represented by a *linear* utility function $U : \mathcal{L} \to \mathbb{R}$: For any $L = [x_1 : p_1, \ldots, x_n : p_n]$,

$$U(L) = p_1 U(x_1) + \cdots p_n U(x_n)$$

We need additional constraints on the decision maker's preferences to rule out preferences over lotteries that are not representable by a linear utility function.

Von Neumann-Morgenstern Theorem



Von Neumann-Morgenstern Representation Theorem Suppose that (P, I) is a rational preference on the set $\mathcal{L}$ of lotteries. Then, (P, I) satisfies Compound Lotteries, Independence and Continuity if, and only if, (P, I) is represented by a **linear utility function**.

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Moreover, the utility function is *unique up to linear transformations*.

Linear Transformations

Suppose that $u : X \to \mathbb{R}$ is a utility function. We say that $u' : X \to \mathbb{R}$ is a **linear transformation of** u provided that there are numbers $a > 0$ and b such that for all $x \in X$: (also called **positive affine transformation**)

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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

	a	b	c	
u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation

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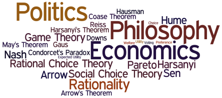

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u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

For all lotteries L and L' and utility functions u,

- if $EU(L,u) > EU(L',u)$ and u' is a linear transformation of u, then $EU(L,u') > EU(L',u')$
- if $EU(L,u) = EU(L',u)$ and u' is a linear transformation of u, then $EU(L,u') = EU(L',u')$

Problems

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- Important issues about how to identify correct descriptions of the outcomes and options.

Allais Paradox

		Red (1)	White (89)	Blue (10)
S_1	A	$1M$	$1M$	$1M$
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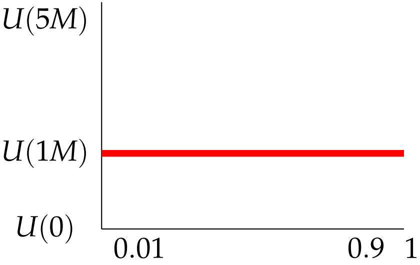

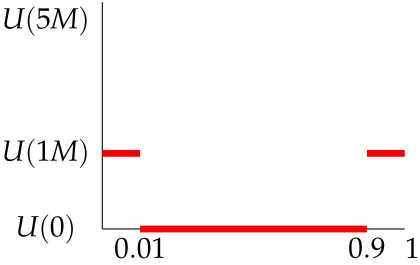

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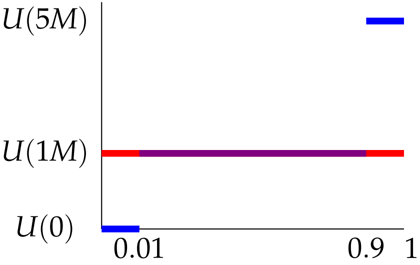

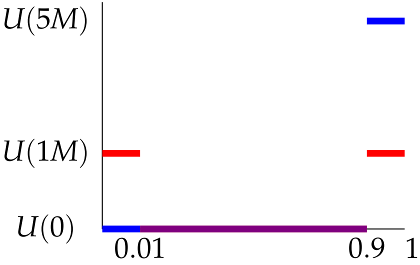

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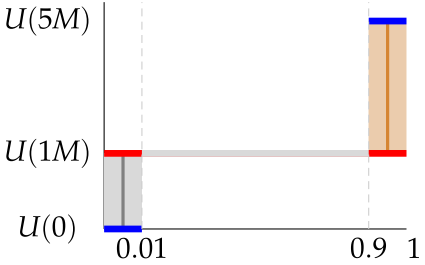

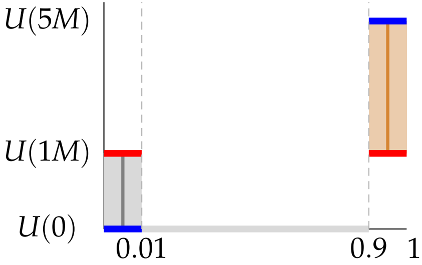

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$A\ P\ B$ if and only if $C\ P\ D$