PHPE 400
Individual and Group Decision Making

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Lotteries

Suppose that $X = \{x_1, \ldots, x_n\}$ is a set of outcomes.

A **lottery** over X is a tuple $[x_1 : p_1, \ldots, x_n : p_n]$ where $\sum_i p_i = 1$.

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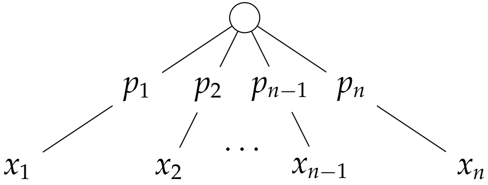

Expected Value of a Lottery

Suppose that the outcomes of a lottery are monetary values. So, $L = [x_1 : p_1, x_2 : p_2, \ldots, x_n : p_n]$, where each x_i is an amount of money. The **expected value** of L is:

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\begin{aligned}
EV([x_1 : p_1, \ldots, x_n : p_n]) &= p_1 \times x_1 + \cdots + p_n \times x_n \\
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E.g., if $L = [\$100 : 0.55, \$50 : 0.25, \$0 : 0.20]$, then

$$EV(L) = 0.55 * 100 + 0.25 * 50 + 0.2 * 0 = 67.5$$

You are given a choice between two lotteries L_1 and L_2. The outcome of the lotteries is determined by flipping a fair coin. The payoff for the two lotteries are given in the following table:

	Heads	Tails
L_1	\$1M	\$1M
L_2	\$3M	\$0

Which of the two lotteries would you choose?

1. L_1
2. L_2
3. I am indifferent between the two lotteries

Problems with using monetary payoffs

- Valuing Money: Doesn't the value of a wager depend on more than merely how much it's expected to pay out? (I.e., your total fortune, how much you personally care about money, etc.). Also, we care about more things than money.

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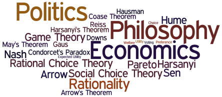

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- The St. Petersburg Paradox: Consider the following wager: I will flip a fair coin until it comes up heads; if the first time it comes up heads is the n^{th} toss, then I will pay you 2^n. What's the most you'd be willing to pay for this wager? What is its expected monetary value?

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- Risk-aversion: Is it irrational to prefer a sure-thing \$x to a wager whose expected payout is \$x?

Expected Utility

Suppose that $X = \{x_1, \ldots, x_n\}$ and $u : X \rightarrow \mathbb{R}$ is a utility function on X.

This can be extended to an expected utility function $EU : \mathcal{L}(X) \rightarrow \mathbb{R}$ where

$$\begin{aligned} EU([x_1 : p_1, \ldots, x_n : p_n], u) &= p_1 \times u(x_1) + \cdots + p_n \times u(x_n) \\ &= \sum_{i=1}^{n} p_i \times u(x_i) \end{aligned}$$

Taking Stock

- Expected value and expected utility (with respect to some utility function) are often used to compare lotteries.
- Comparing lotteries by their expected values may result in a different ranking than comparing lotteries by their expected utility with respect to some utility function.
- To calculate the expected utility of a lottery we need the decision maker's utility function on the outcomes.

Decision Under Certainty: Given any rational preference (P, I) on a set X, there is a utility function $u : X \to \mathbb{R}$ that represents (P, I). That is, a rational decision chooses *as if* she is maximizing her utility.

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What is a **rational** preference over lotteries?

Two preferences

Suppose that $X = \{a, b\}$. Then the set of lotteries over X is

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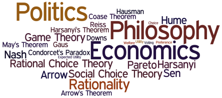

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1. Prefer lotteries that are closer to 50-50: E.g.,

$$[a : \frac{1}{2}, b : \frac{1}{2}] \; P \; [a : \frac{1}{4}, b : \frac{3}{4}] \; I \; [a : \frac{3}{4}, b : \frac{1}{4}] \; P \; [a : 1, b : 0] \; I \; [a : 0, b : 1]$$

2. Prefer lotteries with a higher chance of ending up with a: E.g.,

$$[a : 1, b : 0] \; P \; [a : \frac{3}{4}, b : \frac{1}{4}] \; P \; [a : \frac{1}{2}, b : \frac{1}{2}] \; P \; [a : \frac{1}{4}, b : \frac{3}{4}] \; P \; [a : 0, b : 1]$$

Fact. If (P, I) is a rational preference on $\mathcal{L}$ (plus another condition since $\mathcal{L}$ is infinite), then there is a $U : \mathcal{L} \to \mathbb{R}$ such that $L\ P\ L'$ if and only if $U(L) > U(L')$ and $L\ I\ L'$ if and only if $U(L) = U(L')$.

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$$U_1([a : r, b : (1 - r)]) = -|r - \frac{1}{2}|$$

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The second preference is *rational* while the first preference is irrational: Intuitively, preferences over lotteries should have something to do with preferences over consequences.

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Suppose that the decision maker is rational and has the preference $a \; P \; b$ (the decision maker strictly prefers a to b).

How *should* the decision maker rank the lotteries $L_1 = [a : 0.6, b : 0.4]$ and $L_2 = [a : 0.4, b : 0.6]$?

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4. There is not enough information to answer this question.

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Suppose that the decision maker is rational and has the preference $a \, P \, b$ (the decision maker strictly prefers a to b) and c is another item.

How *should* the decision maker rank the lotteries
$L_1 = [a : 0.6, c : 0.4]$ and $L_2 = [a : 0.6, c : 0.4]$?

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Then, a *rational* decision maker will have the following preferences:

1. The decision maker strictly prefers $[a : 0.6, b : 0.4]$ over $[a : 0.4, b : 0.6]$
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Neither of these preferences can be inferred if all you know is that the decision maker's preferences over lotteries satisfies transitivity and completeness.