PHPE 400
Individual and Group Decision Making

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Utility Profiles

Let X and V be nonempty sets with $|X| \geq 3$ and V finite.

A **utility function** on a set X is a function $u : X \to \mathbb{R}$

Let $\mathcal{U}(X)$ be the set of all functions $u : X \to \mathbb{R}$

A **profile** is a function $\mathbf{U} : V \to \mathcal{U}(X)$, write $\mathbf{U}_i$ for voter i's utility function on X in profile $\mathbf{U}$.

A **Social Welfare Functional (SWFL)** is a function f mapping profiles of utilities to asymmetric relations on X. So for each profile $\mathbf{U}$, $f(\mathbf{U})$ is the social preference order on X.

Sum Utilitarian: Define f_S as follows: For all $x, y \in X$,

$$x\, f_S(\mathbf{U})\, y \text{ if and only if } \sum_i \mathbf{U}_i(x) \geq \sum_i \mathbf{U}_i(y)$$

Lexicographic Maximin: Define f_M as follows: For all $x, y \in X$,

$$x\, f_{LM}(\mathbf{U})\, y \text{ if and only if } \min_i \{\mathbf{U}_i(x)\} \geq \min_i \{\mathbf{U}_i(y)\}$$

breaking ties lexicographically: e.g., $\langle 9, 3, 1, 2 \rangle$ is "less than" $\langle 1, 2, 4, 8 \rangle$.

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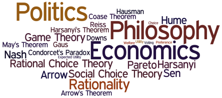
Both SWFLs satisfy versions of Arrow's axioms, including non-dictatorship!

Arrow Axioms



Transitivity/Completeness: For all $\mathbf{U}$ in the domain of f, $f(\mathbf{U})$ is transitive/complete.

Universal Domain: the domain of f is the set of all profiles

Weak Pareto: For all $\mathbf{U}$ in the domain of f, for all $x, y \in X$, if $\mathbf{U}_i(x) > \mathbf{U}_i(y)$ for all $i \in V$, then x is ranked strictly above y according to $f(\mathbf{U})$.

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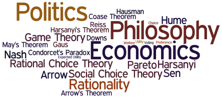

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Independence of Irrelevant Utilities: For all **U** and **U**$'$ in the domain of f, for all $x, y \in X$, if $\mathbf{U}_i(x) = \mathbf{U}'_i(x)$ and $\mathbf{U}_i(y) = \mathbf{U}'_i(y)$ for all $i \in V$, then $x\, f(\mathbf{U})\, y$ if and only if $x\, f(\mathbf{U}')\, y$.

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Why not?

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(*Social Choice and Individual Values*, pp. 10-11).

Linear Transformations

Suppose that $u : X \to \mathbb{R}$ is a utility function. We say that $u' : X \to \mathbb{R}$ is a **linear transformation of** u provided that there are numbers $a > 0$ and b such that for all $x \in X$:

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E.g., suppose that $u : \{a, b, c\} \to \mathbb{R}$ with $u(a) = 3$, $u(b) = 2$ and $u(c) = 0$.

	a	b	c	
u_1	32	22	2	linear transformation
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u_1	32	22	2	linear transformation
u_2	0.75	0.5	0	linear transformation
u_3	9	4	0	not a linear transformation
u_4	-3	-2	0	not a linear transformation

According to standard understanding of utilities in rational choice (as used throughout Economics, Philosophy and Political Science), a decision maker's utility is **unique up to linear transformations**.

U	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
z
x y

U	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
x
y
z

U	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

P	a	b	c
	z	x	y
	x	y	x z
	y	z	

Sum Utilitarian
y
x
z

Equivalent profiles from the vNM perspective

Cardinal measurability equivalence: Given two profiles $\mathbf{U}$ and $\mathbf{U}'$, let $\mathbf{U} \sim_{CM} \mathbf{U}'$ if for every $i \in V$, there are $\alpha_i, \beta_i \in \mathbb{R}$ with $\beta_i > 0$ such that for all $x \in X$, $\mathbf{U}_i(x) = \alpha_i + \beta_i \mathbf{U}'_i(x)$.

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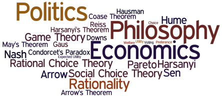

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The following profiles are all cardinal measurability equivalent:

$\mathbf{U}$	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

$\mathbf{U}'$	x	y	z
a	3	1	8
b	300	200	100
c	1	4	1

$\mathbf{U}''$	x	y	z
a	3	1	8
b	300	200	100
c	100	400	100

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This suggests any SWFL should give the same output for any two such profiles:

An Social Welfare Functional f satisfies **CM-invariance** if for all $\mathbf{U}, \mathbf{U}'$, if $\mathbf{U} \sim_{CM} \mathbf{U}'$, then $f(\mathbf{U}) = f(\mathbf{U}')$.

Arrow's theorem

We can now state an update of Arrow's Impossibility Theorem developed by Amartya Sen:

Theorem. Assume $|X| \geq 3$ and that V is finite. If f if an SWFL satisfying **Universal Domain**, **Pareto**, **CM-invariance**, **IIA**, and **Full Rationality**, then f is a *dictatorship*: there is some $i \in V$ such that for all profiles $\mathbf{U}$ and $x, y \in X$, if $\mathbf{U}_i(x) > \mathbf{U}_i(y)$, then $xf(\mathbf{U})y$.

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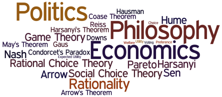

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What now?

In social welfare theory a standard response has been to replace **CM-invariance** by another equivalence relation, assuming a greater degree of *interpersonal comparability of utility*.

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Arrow: "...It requires a definite value judgment not derivable from individual sensations to make the utilities of different individuals dimensionally compatible and still a further value judgment to aggregate them according to any particular mathematical formula. If we look away from the mathematical aspects of the matter, it seems to make no sense to add the utility of one individual, a psychic magnitude in his mind, with the utility of another individual. Even Bentham had his doubts on this point."

(*Social Choice and Individual Values*, p. 11).