PHPE 400
Individual and Group Decision Making

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Grading vs. Ranking

S. Brams and R. Potthoff. *The paradox of grading systems*. Public Choice, 165, pp. 193 - 210, 2015.

Grading vs. Ranking

Suppose that the possible grades are $\{0, 1, \ldots, 20\}$

# of Voters	A	B
1	20	11
1	9	0
1	9	10
Median:	9	10

Majority Judgement Winner: B

Grading vs. Ranking

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Median:	9	10

Majority Judgement Winner: B

2 out of 3 voters prefer A to B

Grading vs. Ranking

Suppose that the possible grades are $\{0, 1, \ldots, 20\}$

# of Voters	A	B
50	20	11
50	9	0
1	9	10
Median:	9	10

Majority Judgement Winner: B

Grading vs. Ranking

Suppose that the possible grades are $\{0, 1, \ldots, 20\}$

# of Voters	A	B
50	20	11
50	9	0
1	9	10
Median:	9	10

Majority Judgement Winner: B

100 out of 101 voters prefer A to B

Grades: $\{0, 1, 2, 3, 4, 5\}$
Candidates: $\{A, B, C\}$
5 Voters

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1	5	0	0
4	0	1	1
Mean:	1	4/5	4/5

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Average Grade Winner: A

Superior Grade Winner: B, C

To conclude, we have identified a paradox of grading systems, which is not just a mirror of the well-known differences that crop up in aggregating votes under ranking systems. Unlike these systems, for which there is no accepted way of reconciling which candidate to choose when, for example, the Hare, Borda and Condorcet winners differ, AV provides a solution when the AG and SG winners differ.

Theorem (Brams and Potthoff). When there are two grades, the AG and SG winners are identical.

The Preference Intensity Problem

51	49
a	b
b	a

51% of the voters have a *slight* preference for a over b and 49% of the voters have a *strong* preference for b over a.

Should candidate a win the election?

The Preference Intensity Problem

80	20
a	b
b	a

80% of the voters *strictly prefer* a over b and 20% of the voters have an "*extremely strong*" preference for b over a.

Should candidate a win the election?

The Preference Intensity Problem

75	25
a	b
b	a

75% of the voters *strictly prefer* a over b and 25% of the voters *strictly prefer* b over a. If a wins, then this will cause harm to the 25% of voters that prefer b to a; and if b wins, this will cause some annoyance to the 75% of the voters that prefer a to b.

How do we weigh the preference of the majority while avoiding harm to the minority?

The Preference Intensity Problem

75	25
a	b
b	a

75% of the voters *strictly prefer* a over b and 25% of the voters *strictly prefer* b over a. If a wins, then this will cause harm to the 25% of voters that prefer b to a; and if b wins, this will cause some annoyance to the 75% of the voters that prefer a to b.

How do we weigh the preference of the majority while avoiding harm to the minority?

- Not all questions should be decided by a vote.
- Education, deliberation, etc. to change the rankings of the enough of the 75% of the voters to ensure that b is the majority opinion.

Systematic Minority

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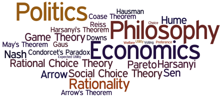
- When preferences are fully polarized and the power of a cohesive majority bloc is secure, the minority remains disenfranchised.

Systematic Minority



- If voters cast a single vote for a single candidate, the majority, no matter how slender, is guaranteed victory.
- When group barriers are permeable, the minority can occasionally belong to the winning side.
- When preferences are fully polarized and the power of a cohesive majority bloc is secure, the minority remains disenfranchised.
- Some solutions:
  - Ensure that the political districts are *fair*: `https://mggg.org/`
  - In some instances power-sharing is imposed directly, and the constitution grants executive positions to specific groups, typically on the basis of their ethnic or religious identity. The problem is that constitutional provisions of this type are difficult to enforce and heavy-handed.

Utility Functions

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A preference ordering is **represented** by a utility function iff x is (weakly) preferred to y provided $u(x) \geq u(y)$

L. Narens and B. Skyrms . *The Pursuit of Happiness Philosophical and Psychological Foundations of Utility*. Oxford University Press, 2020.

Let X and V be nonempty sets with $|X| \geq 3$ and V finite.

Let $\mathcal{U}(X)$ be the set of all functions $u : X \to \mathbb{R}$

A **profile** is a function $\mathbf{U} : V \to \mathcal{U}(X)$, write $\mathbf{U}_i$ for voter i's utility function on X in profile $\mathbf{U}$.

A **Social Welfare Functional (SWFL)** is a function f mapping profiles of utilities to asymmetric relations on X. So for each profile $\mathbf{U}$, $f(\mathbf{U})$ is the social preference order on X.

Sum Utilitarian: Define f_S as follows: For all $x, y \in X$,

$$x \, f_S(\mathbf{U}) \, y \text{ if and only if } \sum_i \mathbf{U}_i(x) \geq \sum_i \mathbf{U}_i(y)$$

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Maximin: Define f_M as follows: For all $x, y \in X$,

$$x \, f_M(\mathbf{U}) \, y \text{ if and only if } \min_i \{\mathbf{U}_i(x)\} \geq \min_i \{\mathbf{U}_i(y)\}$$

$\mathbf{U}$	x	y	z
a	3	1	8
b	3	2	1
c	1	4	1

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a	3	1	8
b	3	2	1
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Sum	7	7	10
Min	1	1	1

- Sum utilitarian: z is ranked above x and y, and x and y are tied.
- Maximin: x, y and z are all tied.

$\mathbf{U}$	x	y	z
a	1	1	8
b	3	1	1
c	5	4	1
Min	1	1	1

Strong Pareto: For all $x, y \in X$, if $\mathbf{U}_i(x) \geq \mathbf{U}_i(y)$ for all $i \in V$ and there is a $j \in V$ such that $U_j(x) > U_j(y)$, then $f(\mathbf{U})$ must rank x strictly above y.

- Maximin violates strong Pareto: x, y and z are all tied; however, shouldn't x be ranked above y?

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- Lexicographic Maximin: rank x above y when $\min_i\{\mathbf{U}_i(x)\} \geq \min_i\{\mathbf{U}_i(y)\}$, breaking ties lexicographically.

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  Lexicographic Maximin: x is ranked above z and z is ranked above y:

$$\langle 1, 3, 5\rangle > \langle 1, 1, 8\rangle > \langle 1, 1, 4\rangle$$

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a	3	1	8
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- Sum utilitarian: z is ranked above x and y, and x and y are tied.
- ~~Maximin: x, y and z are all tied.~~
- Lexicographic Maximin: x is ranked above y, and y is ranked above z.

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Lexicographic Maximin: Define f_M as follows: For all $x, y \in X$,

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breaking ties lexicographically: e.g., $\langle 9, 3, 1, 2\rangle$ is "less than" $\langle 1, 2, 4, 8\rangle$.

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Both SWFLs satisfy versions of Arrow's axioms, including non-dictatorship!