# PHPE 400 <br> Individual and Group Decision Making 

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## Arrow's Theorem

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Let $X$ be a finite set of alternatives with at least three elements and $V$ a finite set of voters.

$$
\text { Social Welfare Function: } f: \mathcal{D} \rightarrow O(X) \text { where } \mathcal{D} \subseteq L(X)^{V}
$$

## Arrow's Theorem

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Social Welfare Function: $f: \mathcal{D} \rightarrow O(X)$ where $\mathcal{D} \subseteq L(X)^{V}$

- For a profile $\mathbf{P}, f(\mathbf{P})$ is the social ranking given $\mathbf{P}$, and we write $a f(\mathbf{P}) b$ when society ranks $a$ at least as high as $b$.
- For a profile $\mathbf{P}$, we write $\mathbf{P}_{i}$ for voter $i^{\prime}$ s ranking.
- $O(X)$ is the set of transitive and complete relations on $X$.
- The set $\mathcal{D}$ is the set of possible elections (the domain of the function $f$ )


## Examples

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Rationality
$\operatorname{Bord} a(\mathbf{P})=\geq_{B c}$ where $a \geq_{B c} b$ provided that the Borda score of $a$ is greater than or equal to the Borda score for $b$.
(Note that $\geq_{B c}$ may not be a linear order)

## Examples


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$\operatorname{Bord} a(\mathbf{P})=\geq_{B c}$ where $a \geq_{B c} b$ provided that the Borda score of $a$ is greater than or equal to the Borda score for $b$.
(Note that $\geq_{B c}$ may not be a linear order)
$\operatorname{Plurality}(\mathbf{P})=\geq_{P l}$ where $a \geq_{P l} b$ provided that the Plurality score of $a$ is greater than or equal to the Plurality score for $b$.
(Note that $\geq_{\text {Pl }}$ may not be a linear order)

## Rankings with Ties

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## Rankings with Ties

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Suppose that $R \in O(X)$ is a ranking with ties on $X$.
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## Rankings with Ties

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Example: Let $X=\{a, b, c\}$ and $R=\{(a, b),(b, a),(a, c),(b, c)\}$. Then:

- $a$ is strictly preferred to $c$
- $b$ is strictly preferred to $c$
- $a$ is not strictly preferred to $b$ (since $a R b$ and $b R a$ )
- $b$ is not strictly preferred to $a$ (since $a R b$ and $b R a$ )


## Universal Domain

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$$
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Voter's are free to choose any ranking, and the voters' choices are independent.

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## Universal Domain

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 Nash Consorcet's Paradox
Rational Choice Theory P ParetoHarsanyi ArrowSocial Choice
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Voter's are free to choose any ranking, and the voters' choices are independent.

The domain of $f$ is the set of all profiles, i.e., $\mathcal{D}=L(X)^{V}$.
If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings. (Arrow 1951 [1963]: 24)

## Pareto/Unanimity

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If each voter ranks $a$ strictly above $b$, then so does the social ranking.

## Pareto/Unanimity

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Rational Choice Theory ParetoHarsany Arrow Social Choice
Rationality
$f: \mathcal{D} \rightarrow O(X)$

If each voter ranks $a$ strictly above $b$, then so does the social ranking.

For all profiles $\mathbf{P} \in \mathcal{D}$ : If $a \mathbf{P}_{i} b$ for each $i \in V$ then $a$ is strictly preferred to $b$ according to $f(\mathbf{P})$

$$
\begin{array}{ccc}
40 & 35 & 25 \\
\hline t & r & k \\
k & k & t \\
r & t & r
\end{array}
$$

According to Plurality, $t$ wins and $k$ loses... even though a majority of voters prefer $k$ to $t$.
$r$ is a spoiler: $r$ splits the vote of all voters rankings $k$ above $t$.


Independence of Irrelevant Alternatives: If $k$ wins and $t$ loses in the profile on the right, then the same should happen in the profile on the left

## Independence of Irrelevant Alternatives

 Nash Rational Choice Theory ParetoHarsany Arrow Rationality

$$
f: \mathcal{D} \rightarrow O(X)
$$

The social ranking (higher, lower, or indifferent) of two alternatives $a$ and $b$ depends only the relative rankings of $a$ and $b$ for each voter.

## Independence of Irrelevant Alternatives

 Nash Rational Choice Theory ParetoHarsany Rrow
Rationality

$$
f: \mathcal{D} \rightarrow O(X)
$$

The social ranking (higher, lower, or indifferent) of two alternatives $a$ and $b$ depends only the relative rankings of $a$ and $b$ for each voter.

For all profiles $\mathbf{P}$ and $\mathbf{P}^{\prime}$ :

$$
\text { If } \mathbf{P}_{i\{a, b\}}=\mathbf{P}_{i\{a, b\}}^{\prime} \text { for all } i \in V \text {, then } f(\mathbf{P})_{\{a, b\}}=f\left(\mathbf{P}^{\prime}\right)_{\{a, b\}} .
$$

where $P_{\{x, y\}}$ is the ranking on $x$ and $y$ defined as follows:

$$
P_{\{x, y\}}=P \cap\{x, y\} \times\{x, y\}
$$

## Independence of Irrelevant Alternatives

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(IIA): For all profiles $\mathbf{P}, \mathbf{P}^{\prime}$ and $x, y \in X$, if $\mathbf{P}_{\{x, y\}}=\mathbf{P}_{\{x, y\}}^{\prime}$, then $f(\mathbf{P})_{\{x, y\}}=f\left(\mathbf{P}^{\prime}\right)_{\{x, y\}}$.

## Independence of Irrelevant Alternatives

 Nash Condorcets Paragox Rational Choice Theory ParetoHarsany ArrowSocial Choice
Rationality
(IIA): For all profiles $\mathbf{P}, \mathbf{P}^{\prime}$ and $x, y \in X$, if $\mathbf{P}_{\{x, y\}}=\mathbf{P}_{\{x, y\}}^{\prime}$, then $f(\mathbf{P})_{\{x, y\}}=f\left(\mathbf{P}^{\prime}\right)_{\{x, y\}}$.
(IIA): For all profiles $\mathbf{P}$ and all $x, y \in X$, if $\mathbf{P}^{\prime}$ is a profile in the domain of $f$ such that $\mathbf{P}_{\{x, y\}}=\mathbf{P}_{\{x, y\}}^{\prime}$, then

- If $x$ defeats $y$ according to $f$ in $\mathbf{P}$, then $x$ defeats $y$ according to $f$ in $\mathbf{P}^{\prime}$
- If $x$ does not defeat $y$ according to $f$ in $\mathbf{P}$, then $x$ does not defeat $y$ according to $f$ in $\mathbf{P}^{\prime}$


## Borda violates IIA, Example 1

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$$
\mathbf{P :} \begin{array}{ccc}
\begin{array}{cc}
45 & 55 \\
a & b \\
c & \\
c & a \\
b & b \\
b & c
\end{array} & \begin{array}{c}
a \\
\\
b
\end{array} & c
\end{array}
$$

$$
\mathbf{P}^{\prime}: \begin{array}{ccc}
\begin{array}{cc}
45 & 55 \\
a & b \\
b & a
\end{array} & & f_{\text {borda }}\left(\mathbf{P}^{\prime}\right) \\
c & a \\
c & c & \\
& c
\end{array}
$$

## Borda violates IIA, Example 1

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$$
\mathbf{P}_{\{\{a, b\}}=\mathbf{P}_{\{\{a, b\}}^{\prime}, \text { but }
$$

$a$ beats $b$ in $\mathbf{P}$ according to Borda, and $b$ beats $a$ in $\mathbf{P}^{\prime}$ according to Borda.

## Borda violates IIA, Example 2

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$$
\text { P: } \begin{array}{ccc} 
& 1 & 1 \\
\hline a & c & \frac{f_{\text {borda }}\left(\mathbf{P}^{\prime}\right)}{a b c} \\
& b & b \\
& c & a \\
& d & d
\end{array}
$$

$\mathbf{P}^{\prime}:$|  | 1 | 1 |  |
| :---: | :---: | :---: | :---: |
|  | $a$ | $c$ | $f_{\text {borda }}\left(\mathbf{P}^{\prime}\right)$ |
|  | $b$ | $b$ | $c$ |
|  | $d$ | $a$ | $d$ |
|  | $c$ | $d$ |  |

## Borda violates IIA, Example 2

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ArrowSocial Choice TheorySen Arrowsocial Rality



$$
\mathbf{P}_{\{\{b, c\}}=\mathbf{P}_{\{\{b, c\}}^{\prime}, \text { but }
$$

$b$ and $c$ are tied in $\mathbf{P}$ according to Borda, and $b$ is ranked above $c$ in $\mathbf{P}^{\prime}$ accodring to Borda.

## Dictatorship

 Arrow Social Choice TheorySen
$f: \mathcal{D} \rightarrow O(X)$

A voter $d \in V$ is a dictator for $f$ if society strictly prefers $a$ over $b$ accodring to $f$ whenever $d$ strictly prefers $a$ over $b$.

## Dictatorship

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$f: \mathcal{D} \rightarrow O(X)$

A voter $d \in V$ is a dictator for $f$ if society strictly prefers $a$ over $b$ accodring to $f$ whenever $d$ strictly prefers $a$ over $b$.

There is a $d \in V$ such that for each profile $\mathbf{P}$, if $a \mathbf{P}_{d} b$ then $a$ is strictly preferred to $b$ according to $f(\mathbf{P})$

Non-Dictatorship: There is no voter that is a dictator for $f$.

## Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies Universal Domain, Independence of Irrelevant Alternatives (IIA) and Pareto is a dictatorship.

- Alternative statement of the theorem: Suppose that there are at least three candidates and finitely many voters. There is no social welfare function that satisfies Universal Domain, Independence of Irrelevant Alternatives (IIA), Pareto, and Non-Dictatorship.


## Evaluative Voting

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In Arrow's theorem, it is assumed that the input is the voters' rankings of the candidates.

One response to Arrow's theorem is to ask for more information from the voters about their opinions of the candidates.

## Approval Voting bridges America's divide.

A simple solution to repair our democracy that is supported by over $70 \%$ of the public!

https:/ /electionscience.org

Approval Voting: Each voter selects a subset of candidates. The candidate with the most "approvals" wins the election.
S. Brams and P. Fishburn. Approval Voting. Birkhauser, 1983.
J.-F. Laslier and M. R. Sanver (eds.). Handbook of Approval Voting. Studies in Social Choice and Welfare, 2010.

