PHPE 400
Individual and Group Decision Making

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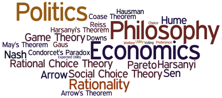
Arrow's Theorem

Let X be a finite set of alternatives with *at least three elements* and V a finite set of voters.

Social Welfare Function: $f : \mathcal{D} \rightarrow O(X)$ where $\mathcal{D} \subseteq L(X)^V$

Arrow's Theorem



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Social Welfare Function: $f : \mathcal{D} \to O(X)$ where $\mathcal{D} \subseteq L(X)^V$

- For a profile $\mathbf{P}$, $f(\mathbf{P})$ is the social ranking given $\mathbf{P}$, and we write $a\, f(\mathbf{P})\, b$ when society ranks a at least as high as b.
- For a profile $\mathbf{P}$, we write $\mathbf{P}_i$ for voter i's ranking.
- $O(X)$ is the set of transitive and complete relations on X.
- The set $\mathcal{D}$ is the set of possible elections (the domain of the function f)

Examples

$Borda(\mathbf{P}) = \geq_{Bc}$ where $a \geq_{Bc} b$ provided that the Borda score of a is greater than or equal to the Borda score for b.

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$Plurality(\mathbf{P}) = \geq_{Pl}$ where $a \geq_{Pl} b$ provided that the Plurality score of a is greater than or equal to the Plurality score for b.

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Rankings with Ties

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Example: Let $X = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, c), (b, c)\}$. Then:

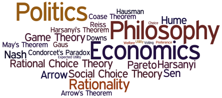
- a is strictly preferred to c
- b is strictly preferred to c
- a is not strictly preferred to b (since $a \mathrel{R} b$ and $b \mathrel{R} a$)
- b is not strictly preferred to a (since $a \mathrel{R} b$ and $b \mathrel{R} a$)

Universal Domain

$f : \mathcal{D} \rightarrow O(X)$

Voter's are free to choose any ranking, and the voters' choices are independent.

Universal Domain



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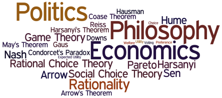
If we do not wish to require any prior knowledge of the tastes of individuals before specifying our social welfare function, that function will have to be defined for every logically possible set of individual orderings. (Arrow 1951 [1963]: 24)

Pareto/Unanimity



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If each voter ranks a strictly above b, then so does the social ranking.

For all profiles $\mathbf{P} \in \mathcal{D}$: If $a\ \mathbf{P}_i\ b$ for each $i \in V$ then
a is strictly preferred to b according to $f(\mathbf{P})$

40	35	25
t	r	k
k	k	t
r	t	r

According to Plurality, t wins and k loses...
even though a majority of voters prefer k to t.

r is a spoiler: r splits the vote of all voters rankings k above t.

40	35	25
t	r	k
k	k	t
r	t	r

40	35	25
t	k	k
k	t	t
r	r	r

Independence of Irrelevant Alternatives: If k wins and t loses in the profile on the right, then the same should happen in the profile on the left

Independence of Irrelevant Alternatives

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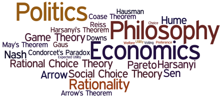
The social ranking (higher, lower, or indifferent) of two alternatives a and b depends only the relative rankings of a and b for each voter.

Independence of Irrelevant Alternatives



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The social ranking (higher, lower, or indifferent) of two alternatives a and b depends only the relative rankings of a and b for each voter.

For all profiles $\mathbf{P}$ and $\mathbf{P}'$:

$$\text{If } \mathbf{P}_{i\{a,b\}} = \mathbf{P}'_{i\{a,b\}} \text{ for all } i \in V\text{, then } f(\mathbf{P})_{\{a,b\}} = f(\mathbf{P}')_{\{a,b\}}.$$

where $P_{\{x,y\}}$ is the ranking on x and y defined as follows:

$$P_{\{x,y\}} = P \cap \{x, y\} \times \{x, y\}$$

Independence of Irrelevant Alternatives

(IIA): For all profiles $\mathbf{P}, \mathbf{P}'$ and $x, y \in X$,
if $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then $f(\mathbf{P})_{\{x,y\}} = f(\mathbf{P}')_{\{x,y\}}$.

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(IIA): For all profiles $\mathbf{P}$ and all $x, y \in X$,
if $\mathbf{P}'$ is a profile in the domain of f such that $\mathbf{P}_{\{x,y\}} = \mathbf{P}'_{\{x,y\}}$, then

- If x defeats y according to f in $\mathbf{P}$, then x defeats y according to f in $\mathbf{P}'$
- If x does not defeat y according to f in $\mathbf{P}$, then x does not defeat y according to f in $\mathbf{P}'$

Borda violates IIA, Example 1

P:

45	55	$f_{borda}(\mathbf{P}')$
a	b	a
c	a	b
b	c	c

P′:

45	55	$f_{borda}(\mathbf{P}')$
a	b	b
b	a	a
c	c	c

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$\mathbf{P}$:	a	b	a
	c	a	b
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	45	55	$f_{borda}(\mathbf{P}')$
$\mathbf{P}'$:	a	b	b
	b	a	a
	c	c	c

$\mathbf{P}_{|\{a,b\}} = \mathbf{P}'_{|\{a,b\}}$, but
a beats b in $\mathbf{P}$ according to Borda, and b beats a in $\mathbf{P}'$ according to Borda.

Borda violates IIA, Example 2

P:

1	1
a	c
b	b
c	a
d	d

$f_{borda}(\mathbf{P}')$
$a\ b\ c$
d

P′:

1	1
a	c
b	b
d	a
c	d

$f_{borda}(\mathbf{P}')$
$a\ b$
c
d

Borda violates IIA, Example 2

P:

1	1
a	c
b	b
c	a
d	d

$f_{borda}(\mathbf{P})$
a b c
d

P′:

1	1
a	c
b	b
d	a
c	d

$f_{borda}(\mathbf{P}')$
a b
c
d

$\mathbf{P}_{|\{b,c\}} = \mathbf{P}'_{|\{b,c\}}$, but

b and c are tied in **P** according to Borda,

and b is ranked above c in **P′** accodring to Borda.

Dictatorship

$f : \mathcal{D} \to O(X)$

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A voter $d \in V$ is a **dictator** for f if society strictly prefers a over b accodring to f *whenever* d strictly prefers a over b.

There is a $d \in V$ such that for each profile $\mathbf{P}$, if $a\ \mathbf{P}_d\ b$ then a is strictly preferred to b according to $f(\mathbf{P})$

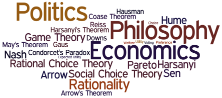
Non-Dictatorship: There is no voter that is a dictator for f.

Arrow's Theorem

Theorem (Arrow, 1951). Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies Universal Domain, Independence of Irrelevant Alternatives (IIA) and Pareto is a dictatorship.

- Alternative statement of the theorem: Suppose that there are at least three candidates and finitely many voters. There is no social welfare function that satisfies Universal Domain, Independence of Irrelevant Alternatives (IIA), Pareto, and Non-Dictatorship.

Evaluative Voting



In Arrow's theorem, it is assumed that the input is the voters' *rankings* of the candidates.

One response to Arrow's theorem is to ask for more information from the voters about their opinions of the candidates.

https://electionscience.org

Approval Voting: Each voter selects a subset of candidates. The candidate with the most "approvals" wins the election.

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds.). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.