PHPE 400
Individual and Group Decision Making

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University of Maryland
pacuit.org

	Plurality	Borda	Ranked Choice	Coombs	Copeland	Minimax	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓

	Plurality	Borda	Ranked Choice	Coombs	Cope-land	Mini-max	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	—	—	—	—	✓	✓	✓
Condorcet Loser	—	✓	✓	✓	✓	—	✓

	Plurality	Borda	Ranked Choice	Coombs	Cope-land	Mini-max	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	–	–	–	–	✓	✓	✓
Condorcet Loser	–	✓	✓	✓	✓	–	✓
Monotonicity	✓	✓	–	–	✓	✓	✓
Positive Involvement	✓	✓	✓	–	–	✓	✓
Multiple Districts	✓	✓	–	–	–	–	–

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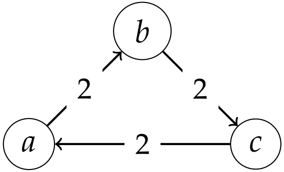
Condorcet Winner	–	–	–	–	✓	✓	✓
Condorcet Loser	–	✓	✓	✓	✓	–	✓
Monotonicity	✓	✓	–	–	✓	✓	✓
Positive Involvement	✓	✓	✓	–	–	✓	✓
Multiple Districts	✓	✓	–	–	–	–	–
Immunity to Spoilers	–	–	–	–	–	✓	✓

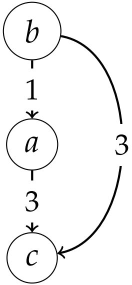
Multiple-Districts Paradox

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

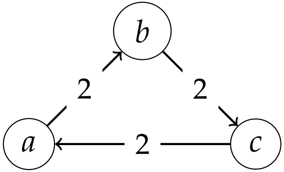
Multiple-Districts Paradox



2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c

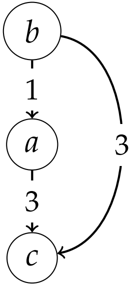


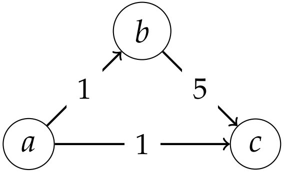
Multiple-Districts Paradox



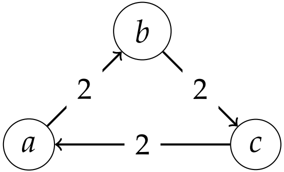
2	2	2
a	b	c
b	c	a
c	a	b

1	2
a	b
b	a
c	c

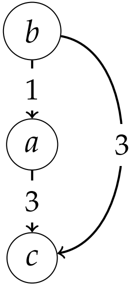


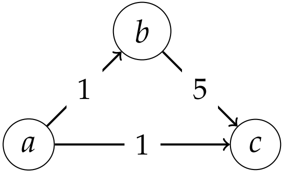


Multiple-Districts Paradox



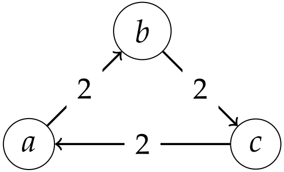
2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c





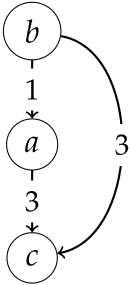
- $\{a, b, c\}$ are the winners in the left profile (assuming Anonymity and Neutrality)
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

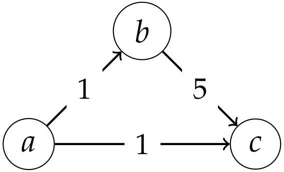
Multiple-Districts Paradox



2	2	2
a	b	c
b	c	a
c	a	b

1	2
a	b
b	a
c	c





- $\{a, b, c\}$ are the winners in the left profile (assuming Anonymity and Neutrality)
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

So, any Condorcet consistent voting method violates the Multiple-Districts Paradox.

Referendum Paradox

D_1	D_2	D_3	D_4	D_5
Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

H. Nurmi (1998). *Voting paradoxes and referenda*. Social Choice and Welfare, Vol. 15, No. 3, pp. 333-350.

H. Dindar, G. Laffond and J. Laine (2017). *The strong referendum paradox*. Quality & Quantity: International Journal of Methodology, 51, pp. 1707 - 1731.

Referendum Paradox

D_1	D_2	D_3	D_4	D_5
Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

- No is the majority outcome overall.

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Referendum Paradox

D_1	D_2	D_3	D_4	D_5
Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

- No is the majority outcome overall.
- Yes wins a majority of the districts: The majority outcome in D_1, D_2, and D_3 is Yes and the majority outcome in D_4 and D_5 is No.

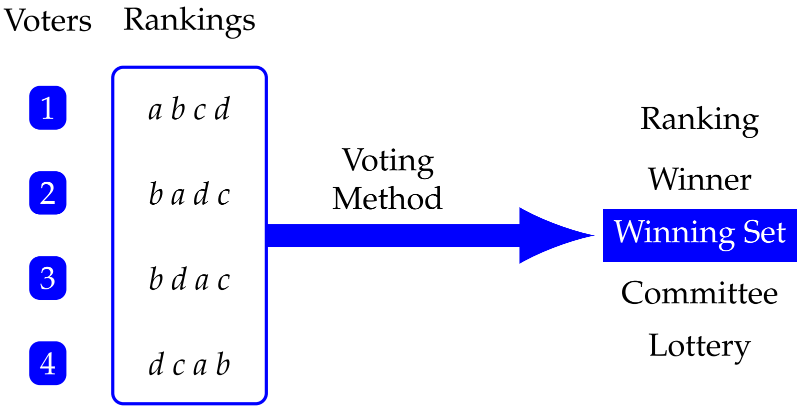
H. Nurmi (1998). *Voting paradoxes and referenda*. Social Choice and Welfare, Vol. 15, No. 3, pp. 333-350.

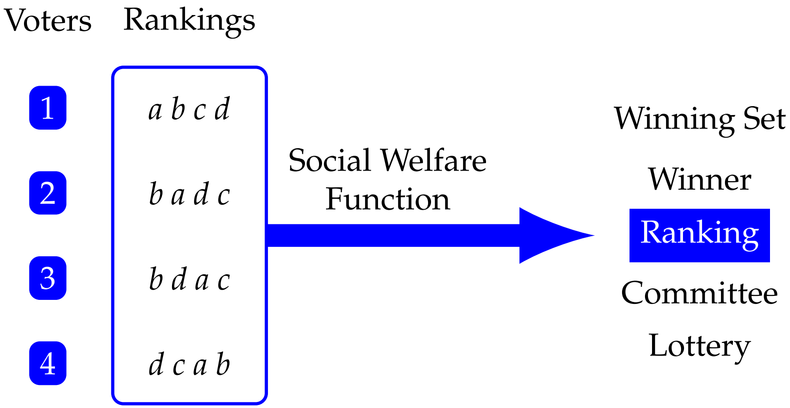
H. Dindar, G. Laffond and J. Laine (2017). *The strong referendum paradox*. Quality & Quantity: International Journal of Methodology, 51, pp. 1707 - 1731.

Electoral College

D. DeWitt and T. Schwartz (2016). *A Calamitous Compact*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 791 - 796.

J. R. Koza (2016). *A Not-So-Calamitous Compact: A Response to DeWitt and Schwartz*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 797 - 804.


Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Voting Method
Ranking
Winner
Winning Set
Committee
Lottery

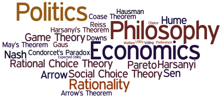

Voters
Rankings
1
2
3
4
a b c d
b a d c
b d a c
d c a b
Social Welfare Function
Winning Set
Winner
Ranking
Committee
Lottery

The Social Choice Model

Notation

- V is a finite set of voters (assume that $V = \{1, 2, 3, \ldots, n\}$)
- X is a (typically finite) set of alternatives, or candidates
- A relation on X is a linear order if it is transitive, irreflexive, and complete (hence, acyclic)
- $L(X)$ is the set of all linear orders over the set X
- $O(X)$ is the set of all reflexive and transitive relations over the set X (i.e., rankings that allow ties)

Notation



- A **profile** for the set of voters V is a sequence of linear orders over X, one for each voter in V.

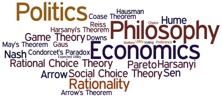
  E.g., $\mathbf{P} = (a\ b\ c, b\ c\ a, c\ a\ b)$ is a profile on three candidates for three voters, the first voter's ranking is $a\ b\ c$ (a is strictly preferred to b and strictly preferred to c and b is strictly preferred to c)

- $L(X)^V$ is the set of all **profiles** for the voters V (similarly for $O(X)^V$)

Preference Aggregation Methods

Social Welfare Function: $f : \mathcal{D} \to O(X)$, where $\mathcal{D} \subseteq L(X)^V$

Preference Aggregation Methods

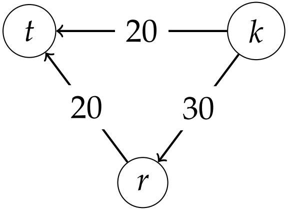


Social Welfare Function: $f : \mathcal{D} \to O(X)$, where $\mathcal{D} \subseteq L(X)^V$

Comments

- $\mathcal{D}$ is the *domain* of the function: it is the set of elections
- Social Welfare Functions are *decisive*: every profile $\mathbf{P}$ in the domain is associated with exactly one ordering over the candidates
- For each profile $\mathbf{P}$, the ranking $f(\mathbf{P})$ is called the **social ordering**

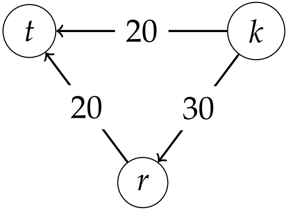
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k\, f(\mathbf{P})\; r\, f(\mathbf{P})\; t$

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

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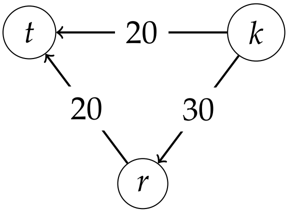

Social Ranking
$k \; r \; t$

Majority Ordering, Copeland, Borda

40	35	25
t	r	k
k	k	r
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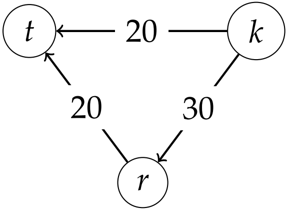

Social Ranking

k r t

k t r

Majority Ordering, Copeland, Borda

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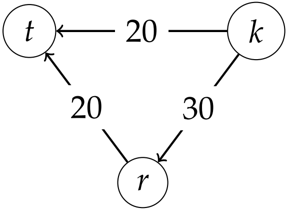
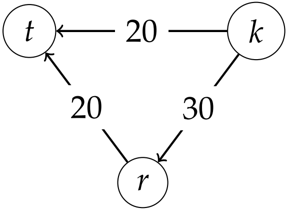


Social Ranking

$k \; r \; t$ Majority Ordering, Copeland, Borda

$k \; t \; r$ Minimize the maximum loss

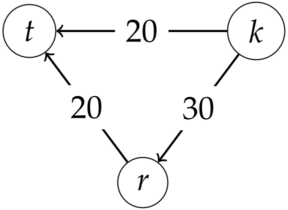
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k \; r \; t$	Majority Ordering, Copeland, Borda
$k \; t \; r$	Minimize the maximum loss
$r \; k \; t$	

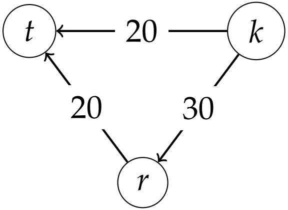
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k \ r \ t$	Majority Ordering, Copeland, Borda
$k \ t \ r$	Minimize the maximum loss
$r \ k \ t$	Instant Runoff

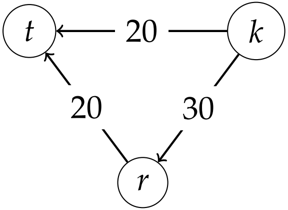
40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k\ r\ t$	Majority Ordering, Copeland, Borda
$k\ t\ r$	Minimize the maximum loss
$r\ k\ t$	Instant Runoff
$t\ r\ k$	

40	35	25
t	r	k
k	k	r
r	t	t



Social Ranking

$k\ r\ t$	Majority Ordering, Copeland, Borda
$k\ t\ r$	Minimize the maximum loss
$r\ k\ t$	Instant Runoff
$t\ r\ k$	Plurality scores

Examples

$Borda(\mathbf{P}) = \geq_{Bc}$ where $a \geq_{Bc} b$ provided that the Borda score of a is greater than or equal to the Borda score for b.

(*Note that $\geq_{Bc}$ may not be a linear order*)

Examples

Borda($\mathbf{P}$) $= \geq_{Bc}$ where $a \geq_{Bc} b$ provided that the Borda score of a is greater than or equal to the Borda score for b.

(*Note that* $\geq_{Bc}$ *may not be a linear order*)

Plurality($\mathbf{P}$) $= \geq_{Pl}$ where $a \geq_{Pl} b$ provided that the Plurality score of a is greater than or equal to the Plurality score for b.

(*Note that* $\geq_{Pl}$ *may not be a linear order*)

Examples

$Borda(\mathbf{P}) \;= \geq_{Bc}$ where $a \geq_{Bc} b$ provided that the Borda score of a is greater than or equal to the Borda score for b.

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(*Note that* $\geq_{Pl}$ *may not be a linear order*)

$Maj(\mathbf{P}) = \; >^M_{\mathbf{P}}$ where $a >^M_{\mathbf{P}} b$ provided that $Margin_{\mathbf{P}}(a, b) > 0$

(*Problem:* $>^M_{\mathbf{P}}$ *may not be transitive*)

Arrow's Theorem

Let X be a finite set with *at least three elements* and V a finite set of n voters.

Social Welfare Function: $f : \mathcal{D} \to O(X)$ where $\mathcal{D} \subseteq L(X)^V$

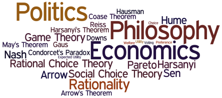
Arrow's Theorem

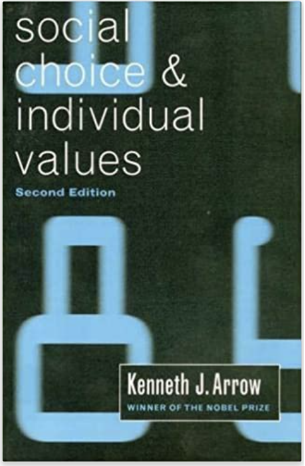
Let X be a finite set with *at least three elements* and V a finite set of n voters.

Social Welfare Function: $f : \mathcal{D} \rightarrow O(X)$ where $\mathcal{D} \subseteq L(X)^V$

- For a profile $\mathbf{P}$, $f(\mathbf{P})$ is the social ranking given $\mathbf{P}$, and we write $a\, f(\mathbf{P})\, b$ when society ranks a at least as high as b.
- For a profile $\mathbf{P}$, we write $\mathbf{P}_i$ for voter i's ranking.
- $O(X)$ is the set of transitive and complete relations on X.

Arrow’s Impossibility Theorem

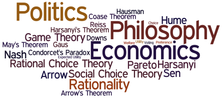


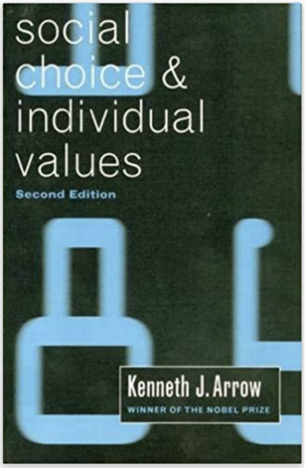


“For an area of study to become a recognized field, or even a recognized subfield, two things are required: It must be seen to have coherence, and it must be seen to have depth. The former often comes gradually, but the latter can arise in a single flash of brilliance....With social choice theory, there is little doubt as to the seminal result that made it a recognized field of study: Arrow’s impossibility theorem.”

A. Taylor, Social Choice and the Mathematics of Manipulation

Arrow's Impossibility Theorem





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