PHPE 400
Individual and Group Decision Making

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Do the previous arguments for majority rule apply when there are more than 2 candidates? No!

- ✓ Group decision problems often exhibit a *combinatorial structure*. For example, voting on a number of yes/no issues in a referendum, or voting on different interconnected issues.
- ▶ As we have seen, there are many reasonable voting methods that generalize Majority Rule for more than 2 candidates. Is there a voting method that satisfies *all* principles of group decision making?

Principles of group decision making

- **Anonymity**: If voters swap their ballots, then the outcome is unaffected.
- **Neutrality**: If candidates are exchanged in every ranking, then the outcome changes accordingly.
- **Resoluteness**: Always elect a single winner.

Condorcet Triples and Resoluteness

n	n	n
a	b	c
b	c	a
c	a	b

n	n	n
a	c	b
c	b	a
b	a	c

Fact. In both profiles, any voting method satisfying anonymity and neutrality must select all candidates as winners

1	1	1
a	b	c
b	c	a
c	a	b

Consider $\mathbf{P} = (a\ b\ c, b\ c\ a, c\ a\ b)$ and suppose that $F(a\ b\ c, b\ c\ a, c\ a\ b) = \{a\}$

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1. Swap a and b in everyone's rankings in the given profile. Then, by Neutrality:

$$F(b\ a\ c, a\ c\ b, c\ b\ a) = \{b\}$$

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$$F(b\ a\ c, a\ c\ b, c\ b\ a) = \{b\}$$

2. Swap b and c in everyone's rankings in the profile from step 1. Then, by Neutrality:
$$F(c\ a\ b, a\ b\ c, b\ c\ a) = \{c\}$$

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2. Swap b and c in everyone's rankings in the profile from step 1. Then, by Neutrality:
$$F(c\ a\ b, a\ b\ c, b\ c\ a) = \{c\}$$

3. By Anonymity, the original profile and the profile in step 3 must have the same winners:

$$F(a\ b\ c, b\ c\ a, c\ a\ b) = F(c\ a\ b, a\ b\ c, b\ c\ a)$$

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4. 1 and 2 contradict 3 since
$F(a\ b\ c, b\ c\ a, c\ a\ b) = \{a\} \neq \{c\} = F(c\ a\ b, a\ b\ c, b\ c\ a)$.

So, tie-breaking cannot be built-in to a voting method: there is no voting method that satisfies Anonymity, Neutrality and always elects a single winner.

Recall Weak Positive Responsiveness

- F satisfies **weak positive responsiveness** if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, if

  1. $a \in F(\mathbf{P})$ (a is a winner in $\mathbf{P}$ according to F) and
  2. $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter who ranked a uniquely last in $\mathbf{P}$ switching to ranking a uniquely first in $\mathbf{P}'$,

  then $F(\mathbf{P}') = \{a\}$ (a is the **unique** winner in $\mathbf{P}'$ according to F).

Monotonicity

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More-is-Less Paradox: If a candidate *c* is elected under a given a profile of rankings of the competing candidates, it is possible that, *ceteris paribus*, *c* may not be elected if some voter(s) raise *c* in their rankings.

P. Fishburn and S. Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

More-is-Less Paradox: Ranked Choice

6	5	4	2
a	c	b	b
b	a	c	a
c	b	a	c

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a	c	b	a
b	a	c	b
c	b	a	c

More-is-Less Paradox: Ranked Choice

6	5	4	2
a	*c*	*b*	*b*
b	*a*	*c*	*a*
c	*b*	*a*	*c*

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b	*a*	*c*	*b*
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6	5	4	2
a	*c*	*b*	*b*
b	*a*	*c*	*a*
c	*b*	*a*	*c*

Ranked Choice Winner: *a*

6	5	4	2
a	*c*	*b*	*a*
b	*a*	*c*	*b*
c	*b*	*a*	*c*

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a	c	b	b
b	a	c	a
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Ranked Choice Winner: a

6	5	4	2
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Ranked Choice Winner: c

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More on Monotonicity

Key idea: Unequivocal increase in support for a candidate should not result in that candidate going from being a winner to being a loser.

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1. *monotonicity*: if a candidate x is a winner given a preference profile $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter moving x up in their ranking, then x should still be a winner given $\mathbf{P}'$.

Positive and Negative Involvement

Consider the following perverse responses, dubbed Strong No Show Paradoxes (cf. Fishburn and Brams 1983), when a coalition C of voters comes to the polls:

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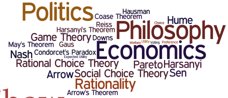
2. Had the voters in C stayed home, candidate a would have lost; and everyone in C ranked a last; but this caused a to *win*.

Positive and Negative Involvement



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Following Saari (1995), we call 1 a violation of Positive Involvement and 2 a violation of Negative Involvement.

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People are often shocked to learn that these are possible with standard voting methods: **Instant Runoff** violates Negative Involvement, while some Condorcet methods violate both versions.

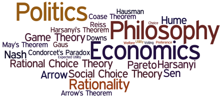
Violations of Negative Involvement

Remarkably, in the 2022 Alaska election in which Peltola won, removing anywhere between 5,170 and 8,406 voters with the ranking

$$\textit{Palin} > \textit{Begich} > \textit{Peltola}$$

leads to Begich winning, so by ranking Peltola *last*, they “caused” her to win!

Violations of Negative Involvement



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For details, see https://github.com/voting-tools/election-analysis or Smith and Navratil’s (2022) paper, “If Peltola had more votes, she would have lost.”

Coombs violates Positive Involvement

2	2	1	1	2	1	1
c	b	d	d	c	a	b
a	a	c	a	b	d	d
b	c	b	c	d	b	a
d	d	a	b	a	c	c

Coombs winner: $\{b\}$

(the order of elimination is d, c)

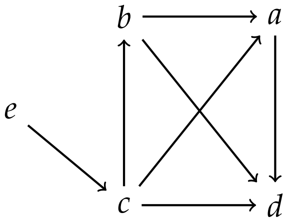
2	2	1	1	2	1	1	1
c	b	d	d	c	a	b	b
a	a	c	a	b	d	d	d
b	c	b	c	d	b	a	c
d	d	a	b	a	c	c	a

Coombs winner: $\{c\}$

(a and d are tied for the most last place votes)

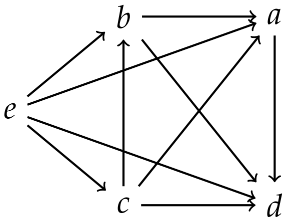
Copeland violates Positive Involvement

2	1	1
e	c	a
c	b	d
b	a	b
a	d	e
d	e	c



Copeland winners: $\{c\}$

2	1	1	1
e	c	a	c
c	b	d	e
b	a	b	d
a	d	e	c
d	e	c	a



Copeland winners: $\{e\}$

More Principles

Pareto/Unanimity: In any profile **P**, if every voter ranks x strictly above y, then y is not a winner.

Every voting method we have studied satisfies Pareto.

More Principles

Condorcet: In any profile **P**, if x is a Condorcet winner, then x is the unique winner.

Condorcet Loser: In any profile **P**, if x is a Condorcet loser, then x is not a winner.

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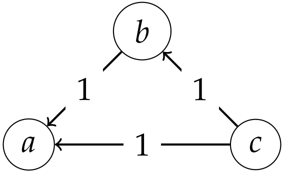
Plurality violates both the Condorcet Winner and Condorcet Loser principles.

2	2	2	1
c	b	a	a
b	c	c	b
a	a	b	c



Plurality Winners: $\{a\}$
Condorcet Winner: c
Condorcet Loser: a

Multiple-Districts Paradox

Multiple-Districts: If a candidate wins in each district, then that candidate should also win when the districts are merged.

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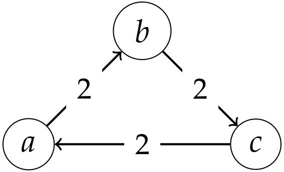
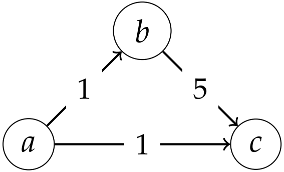


2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c



- $\{a, b, c\}$ are the winners in the left profile (assuming Anonymity and Neutrality)
- b is the Condorcet winner in the right profile
- a is the Condorcet winner in the combined profiles

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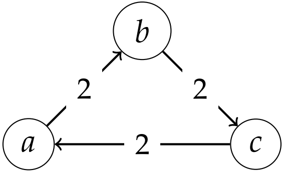
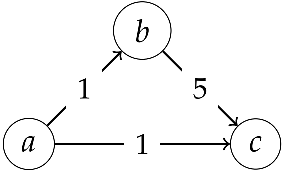


2	2	2	1	2
a	b	c	a	b
b	c	a	b	a
c	a	b	c	c



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So, any Condorcet consistent voting method violates the Multiple-Districts Paradox.

Referendum Paradox

D_1	D_2	D_3	D_4	D_5
Yes	Yes	No	No	No
No	Yes	Yes	No	No
Yes	No	Yes	No	No

H. Nurmi (1998). *Voting paradoxes and referenda*. Social Choice and Welfare, Vol. 15, No. 3, pp. 333-350.

H. Dindar, G. Laffond and J. Laine (2017). *The strong referendum paradox*. Quality & Quantity: International Journal of Methodology, 51, pp. 1707 - 1731.

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- No is the majority outcome overall.

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Yes	Yes	No	No	No
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Yes	No	Yes	No	No

- No is the majority outcome overall.
- Yes wins a majority of the districts: The majority outcome in D_1, D_2, and D_3 is Yes and the majority outcome in D_4 and D_5 is No.

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Gerrymandering

https://mggg.org/

Electoral College

D. DeWitt and T. Schwartz (2016). *A Calamitous Compact*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 791 - 796.

J. R. Koza (2016). *A Not-So-Calamitous Compact: A Response to DeWitt and Schwartz*. Political Science & Politics, Volume 49, Special Issue 4: Elections in Focus, pp. 797 - 804.

Principles

Anonymity: If voters swap their ballots, then the outcome is unaffected.

Neutrality: If candidates are exchanged in every ranking, then the outcome changes accordingly.

Pareto: If every voter ranks a strictly above b (i.e., b is *dominated* by a) then b is not a winner.

Condorcet: When the Condorcet winner exists, then it is the unique winner.

Condorcet Loser: Do not elect the Condorcet loser whenever it exists.

Principles

Monotonicity: if a candidate x is a winner given a preference profile $\mathbf{P}$, and $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter moving x up in their ranking, then x should still be a winner given $\mathbf{P}'$.

Positive Involvement: if a candidate x is a winner given $\mathbf{P}$, and $\mathbf{P}^*$ is obtained from $\mathbf{P}$ by adding a new voter who ranks x in first place, then x should still be a winner given $\mathbf{P}^*$.

Multiple-Districts: Suppose that a voting population is divided into districts. If a candidate wins in each district, then that candidate should also win when the districts are merged.

Is there a voting method that satisfies *all* of them?

	Plurality	Borda	Ranked Choice	Coombs	Copeland	Minimax	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓

	Plurality	Borda	Ranked Choice	Coombs	Cope-land	Mini-max	Split Cycle
Anonymity	✓	✓	✓	✓	✓	✓	✓
Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	—	—	—	—	✓	✓	✓
Condorcet Loser	—	✓	✓	✓	✓	—	✓

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Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	—	—	—	—	✓	✓	✓
Condorcet Loser	—	✓	✓	✓	✓	—	✓
Monotonicity	✓	✓	—	—	✓	✓	✓
Positive Involvement	✓	✓	✓	—	—	✓	✓
Multiple Districts	✓	✓	—	—	—	—	—

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Neutrality	✓	✓	✓	✓	✓	✓	✓
Pareto	✓	✓	✓	✓	✓	✓	✓
Condorcet Winner	–	–	–	–	✓	✓	✓
Condorcet Loser	–	✓	✓	✓	✓	–	✓
Monotonicity	✓	✓	–	–	✓	✓	✓
Positive Involvement	✓	✓	✓	–	–	✓	✓
Multiple Districts	✓	✓	–	–	–	–	–
Immunity to Spoilers	–	–	–	–	–	✓	✓