PHPE 400
Individual and Group Decision Making

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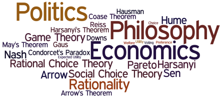
When there are only two options, can we argue that majority rule is the "best" procedure?

Setting aside the possibility of using lotteries, May's Theorem is a proceduralist justification of majority rule showing that it is the unique procedure satisfying normative principles of group decision making.

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May’s Theorem: Details



Voters: $V = \{1, 2, 3, \ldots, n\}$ is the set of n voters.

Candidates: $X = \{a, b\}$ is set of candidates.

Suppose that voters can submit one of 3 rankings:

1. $a\ P\ b$: a is ranked above b (“vote for a”)
2. $a\ I\ b$: a and b are tied (“vote for a and b”)
3. $b\ P\ a$: b is ranked above a (“vote for b”)

Note that $a\ I\ b$ and $b\ I\ a$ is the same ballot since indifference is symmetric.

Let $O(X)$ be the set of 3 rankings on X.

May's Theorem: Details

The set of **profiles** is $O(X)^V$, where a profile assigns to each voter one of the three rankings from $O(X)$.

Given a profile $\mathbf{P} \in O(X)^V$ and a voter $i \in V$, we write $\mathbf{P}_i$ for the ranking of voter i.

E.g., suppose that $V = \{1, 2, 3, 4\}$ and consider the profile

$$\mathbf{P} = (a\ P\ b, a\ I\ b, b\ P\ a, a\ P\ b)$$

Then, $\mathbf{P}_2$ is the ranking $a\ I\ b$ (voter 2 is indifferent between a and b).

May's Theorem: Details

Social Choice Function: $F : O(X)^V \to \wp(X)$.

Where for all profiles $\mathbf{P}$ from $O(X)^V$, $F(\mathbf{P})$ is the set of winners.

We assume that for all profile $\mathbf{P}$, $F(\mathbf{P}) \neq \varnothing$ (so there is always at least one winner).

Anonymity and Neutrality

- F satisfies **anonymity**: permuting the voters does not change the set of winners.
- F satisfies **neutrality**: permuting the candidates results in a winning set that is permuted in the same way.

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$\Longrightarrow$ in 2-candidate profiles, if the same number of voters rank $\boldsymbol{a}$ above $\boldsymbol{b}$ as $\boldsymbol{b}$ above $\boldsymbol{a}$, then $\boldsymbol{a} \in F(\mathbf{P})$ if, and only if, $\boldsymbol{b} \in F(\mathbf{P})$

($\boldsymbol{a}$ wins according to F if and only if $\boldsymbol{b}$ wins according to F).

Weak Positive Responsiveness

- F satisfies **weak positive responsiveness** if for any profiles $\mathbf{P}$ and $\mathbf{P}'$, if

  1. $a \in F(\mathbf{P})$ (a is a winner in $\mathbf{P}$ according to F) and
  2. $\mathbf{P}'$ is obtained from $\mathbf{P}$ by one voter who ranked a uniquely last in $\mathbf{P}$ switching to ranking a uniquely first in $\mathbf{P}'$,

  then $F(\mathbf{P}') = \{a\}$ (a is the **unique** winner in $\mathbf{P}'$ according to F).

Profile	Voter 1	Always a	Minority	Consensus	Majority
$(a\ P\ b, a\ P\ b)$	a	a	b	a	a
$(a\ P\ b, a\ I\ b)$	a	a	b	a, b	a
$(a\ P\ b, b\ P\ a)$	a	a	a, b	a, b	a, b
$(a\ I\ b, a\ P\ b)$	a, b	a	b	a, b	a
$(a\ I\ b, a\ I\ b)$	a, b	a	a, b	a, b	a, b
$(a\ I\ b, b\ P\ a)$	a, b	a	a	a, b	b
$(b\ P\ a, a\ P\ b)$	b	a	a, b	a, b	a, b
$(b\ P\ a, a\ I\ b)$	b	a	a	a, b	b
$(b\ P\ a, b\ P\ a)$	b	a	a	b	b

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$(a\ I\ b, a\ P\ b)$	a, b	a	b	a, b	a
$(a\ I\ b, a\ I\ b)$	a, b	a	a, b	a, b	a, b
$(a\ I\ b, b\ P\ a)$	a, b	a	a	a, b	b
$(b\ P\ a, a\ P\ b)$	b	a	a, b	a, b	a, b
$(b\ P\ a, a\ I\ b)$	b	a	a	a, b	b
$(b\ P\ a, b\ P\ a)$	b	a	a	b	b

	Anonymity	Neutrality	Positive Responsiveness
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Always a			
Minority			
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Majority			

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$(a\ P\ b, b\ P\ a)$	a	a	a, b	a, b	a, b
$(a\ I\ b, a\ P\ b)$	a, b	a	b	a, b	a
$(a\ I\ b, a\ I\ b)$	a, b	a	a, b	a, b	a, b
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$(b\ P\ a, a\ I\ b)$	b	a	a	a, b	b
$(b\ P\ a, b\ P\ a)$	b	a	a	b	b

	Anonymity	Neutrality	Weak Positive Responsiveness
Voter 1	✗		
Always a	✓		
Minority	✓		
Consensus	✓		
Majority	✓		

Profile	Voter 1	Always a	Minority	Consensus	Majority
$(a\ P\ b, a\ P\ b)$	a	a	b	a	a
$(a\ P\ b, a\ I\ b)$	a	a	b	a, b	a
$(a\ P\ b, b\ P\ a)$	a	a	a, b	a, b	a, b
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$(a\ I\ b, a\ I\ b)$	a, b	a	a, b	a, b	a, b
$(a\ I\ b, b\ P\ a)$	a, b	a	a	a, b	b
$(b\ P\ a, a\ P\ b)$	b	a	a, b	a, b	a, b
$(b\ P\ a, a\ I\ b)$	b	a	a	a, b	b
$(b\ P\ a, b\ P\ a)$	b	a	a	b	b

	Anonymity	Neutrality	Positive Responsiveness
Voter 1	✗	✓	
Always a	✓	✗	
Minority	✓	✓	
Consensus	✓	✓	
Majority	✓	✓	

	Profile	Voter 1	Always a	Minority	Consensus	Majority
$\mathbf{P}'$	$(a\ P\ b, a\ P\ b)$	a	a	b	a	a
	$(a\ P\ b, a\ I\ b)$	a	a	b	a, b	a
$\mathbf{P}$	$(a\ P\ b, b\ P\ a)$	a	a	a, b	a, b	a, b
$\mathbf{P}'$	$(a\ I\ b, a\ P\ b)$	a, b	a	b	a, b	a
	$(a\ I\ b, a\ I\ b)$	a, b	a	a, b	a, b	a, b
$\mathbf{P}$	$(a\ I\ b, b\ P\ a)$	a, b	a	a	a, b	b
$\mathbf{P}'$	$(b\ P\ a, a\ P\ b)$	b	a	a, b	a, b	a, b
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Always a	✓	✗	✓
Minority	✓	✓	✗
Consensus	✓	✓	✗
Majority	✓	✓	✓

	Anonymity	Neutrality	Weak Positive Responsiveness
Voter 1	✗	✓	✗
Always a	✓	✗	✓
Minority	✓	✓	✗
Consensus	✓	✓	✗
Majority	✓	✓	✓

May's Theorem

Theorem (May 1952)

Let F be a voting method on the domain of two-alternative profiles. Then the following are equivalent:

1. F satisfies anonymity, neutrality, and weak positive responsiveness;
2. F is majority voting.

Proof Sketch

Suppose that F satisfies Anonymity, Neutrality and Positive Responsiveness. Can we have $F(a\ P\ b, a\ P\ b, b\ P\ a) = \{b\}$?

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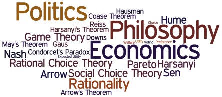

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Suppose that $F(a\ P\ b, a\ P\ b, b\ P\ a) = \{b\}$

By Neutrality, $F(b\ P\ a, b\ P\ a, a\ P\ b) = \{a\}$

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Contradiction: Since F is a function, we can't have $F(a\ P\ b, a\ P\ b, b\ P\ a) = \{b\}$ and $F(a\ P\ b, a\ P\ b, b\ P\ a) = \{a\}$

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Other characterizations

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pp. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pp. 89 - 94, 2003.

May’s Theorem is a *proceduralist* justification of majority rule showing that Majority Rule is the unique group decision method satisfying two basic principles of fairness (Anonymity and Neutrality) and a basic principle ensuring that the outcome responds appropriately to the voters’ opinions (Weak Positive Responsiveness).

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We can also give an *epistemic* justification of majority rule showing that has a high probability of identifying the correct answer to a question.

Epistemic Justification of Majority Rule

In many group decision making problems, one of the alternatives is the *correct* one. Which group decision making method is best for finding the "correct" alternative?

The Condorcet Jury Theorem

https://cjt-tutorial.streamlit.app/

Condorcet Jury Theorem

- $V = \{1, 2, \ldots, n\}$ is the set of experts.
- $\{0, 1\}$ is the set of outcomes.
- $\mathbf{x}$ be a random variable (called the **state**) whose values range over the two outcomes. We write $\mathbf{x} = 1$ when the outcome is 1 and $\mathbf{x} = 0$ when the outcome is 0.
- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are random variables representing the votes for experts $1, 2, \ldots, n$. For each $i = 1, \ldots, n$, we write $\mathbf{v}_i = 1$ when expert i's vote is 1 and $\mathbf{v}_i = 0$ when expert i's vote is 0.
- R_i is the event that expert i votes correctly: it is the event that $\mathbf{v}_i$ coincides with $\mathbf{x}$ (i.e., $\mathbf{v}_i = 1$ and $\mathbf{x} = 1$ or $\mathbf{v}_i = 0$ and $\mathbf{x} = 0$).

Condorcet Jury Theorem

Independence: The correctness events $R_1, R_2, \ldots, R_n$ are independent.

Competence: The experts' competences $Pr(R_i)$ (i) exceeds $\frac{1}{2}$ and (ii) is the same for each voter i.

Condorcet Jury Theorem: Assume Independence and Competence. Then, as the group size increases, the probability of that the majority is correct (i) increases (growing reliability), and (ii) tends to one (infallibility).

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The Condorcet Jury Theorem is an *epistemic* justification of majority rule showing that under the assumption that the voters are *competent* in the sense that each voters has a greater than 50% chance of voting correctly and that the events that the voters are correct are independent, then the probability that the majority is correct increases to 1 as the size of the group increases.

Can May's Theorem be generalized to more than 2 candidates?

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- Group decision problems often exhibit a *combinatorial structure*. For example, voting on a number of yes/no issues in a referendum, or voting on different interconnected issues, or selecting a committee from a set of candidates.
- As we have seen, there are many reasonable voting methods that generalize Majority Rule for more than 2 candidates.